1. Mayer Vietoris

2. $X \overset{h.e.}{\simeq} S^1$

3. Definition of det: $T_I \det(A) = \operatorname{Tr}(A)$.

4. $\partial M \xrightarrow{i} M \xrightarrow{F} \mathbb{R}^n - \{0\} \xrightarrow{\frac{x}{|x|}} S^{n-1} \xrightarrow{f^{-1}} \partial M$ is identity

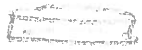
5. Exterior derivative, Stokes

6. Consider the image of the curve as projected on the upper hemisphere of S^2.
It must be smooth at $(1,0,0)$.

7. Lifting criterion, one-to-one correspondence between connected covers and subgroups of π_1, if $\tilde{N}$ is non-compact, $\deg \tilde{f} = 0$, a contradiction as $\deg f \neq 0$.

Geo/Top Fall 07:

① X path-connected, $H_i(X)=0$ for all $0<i\leq n$.
Compute $H_i(X\times S^n)$ for all i, $0<i\leq n$

Let $A = X\times$ , $B = X\times$  so that
$A\cap B = X\times$  $\simeq X\times S^{n-1}$, and $A\cup B = X\times S^n$.

Now apply Mayer-Vietoris:

• Case $i>1$:

$$\cdots\to H_i(A)\oplus H_i(B)\to H_i(X\times S^n)\to H_{i-1}(A\cap B)$$
$$\to H_{i-1}(A)\oplus H_{i-1}(B)\to\cdots$$

$$\Rightarrow \cdots\to \underbrace{H_i(X)\oplus H_i(X)}_{0}\to H_i(X\times S^n)\to H_{i-1}(X\times S^{n-1})$$
$$\to \underbrace{H_{i-1}(X)\oplus H_{i-1}(X)}_{0} \Rightarrow \underline{H_i(X\times S^n)\cong H_{i-1}(X\times S^{n-1})}$$ by exactness.

Case $i=1$:

$$\cdots\to 0\to H_1(X\times S^n)\to H_0(X\times S^{n-1})\xrightarrow{(i^*,j^*)} H_0(X)\oplus H_0(X)$$
$$\to H_0(X\times S^n)\to 0$$; X path connected, so $H_0(X)\cong H_0(X\times S^n)\cong\mathbb{Z}$,

so we have:

$$\cdots\to 0\to H_1(X\times S^n)\to\mathbb{Z}\xrightarrow{(i^*,j^*)}\mathbb{Z}\oplus\mathbb{Z}\to\mathbb{Z}\to 0.$$

Now consider the inclusions:

$i: A\cap B\hookrightarrow A$

$X\times$  $\hookrightarrow X\times$ 

$j: A\cap B\hookrightarrow B$

$X\times$  $\hookrightarrow X\times$ 

$\rightsquigarrow$ $i^*: H_0(A\cap B)\to H_0(A)$
$\alpha\longmapsto\alpha$

$j^*: H_0(A\cap B)\to H_0(B)$
$\alpha\longmapsto\alpha$

Clearly injective since path connected & takes vertex to vertex.

So we've shown that $(i^*, j^*)_0 : H_0(X \times S^{n-1}) \to H_0(X) \oplus H_0(X)$ is injective, hence:

$$0 \to H_1(X \times S^n) \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{(i^*,j^*)_0} \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \xrightarrow{\partial_0} 0$$

$$\ker (i^*, j^*)_0 = 0 \Rightarrow \operatorname{Im} \partial_1 = 0 \Rightarrow H_1(X \times S^n) = 0$$

since ∂_1 injective by exactness

$$\text{So } \begin{cases} H_0(X \times S^n) = \mathbb{Z} \\ H_1(X \times S^n) = 0 \\ H_i(X \times S^n) \cong H_i(X \times S^{n-1}) & i > 1. \end{cases}$$

(2.) C_1, C_2 disjoint circles in $\mathbb{R}^3$, $A = S^1 \times [0,1]$ cylinder

Let $X = \mathbb{R}^3 \amalg A$ via gluing ∂A to C_1 & C_2 by homeomorphisms:

Compute $\pi_1(X)$.

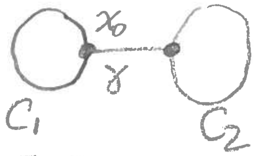
Consider curve γ connecting C_1 & C_2

Now let $A_1 = A \cup C_1 \cup C_2 \cup \gamma$ and let $A_2 = \mathbb{R}^3$

Then $A_1 \cup A_2 = X$ and $A_1 \cap A_2 = C_1 \cup C_2 \cup \gamma$, which is connected, hence we may apply Seifert – Van Kampen:

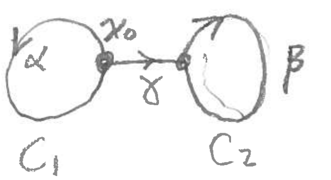
$$\pi_1(X) \cong \pi_1(A_1) * \pi_1(A_2) \big/ \langle i_{12}(\omega)\, i_{21}(\omega)^{-1} \rangle$$

where $i_{12} : \pi_1(A_1 \cap A_2) \to \pi_1(A_1)$
$i_{21} : \pi_1(A_1 \cap A_2) \to \pi_1(A_2)$ are induced by inclusion

See that $A_1 \cap A_2 = C_1 \cup C_2 \cup \gamma =$ 

, hence $\pi_1(A_1 \cap A_2) = \mathbb{Z} * \mathbb{Z}$

with generators the loops α, $\gamma\beta\bar{\gamma}$ (based at x_0):

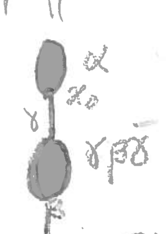
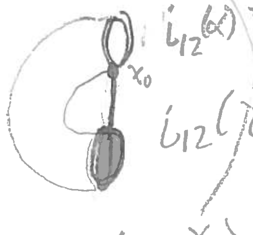


So we need only consider $i_{12}(\alpha)$, $i_{12}(\gamma\beta\bar{\gamma})$, $i_{21}(\alpha)$, $i_{21}(\gamma\beta\bar{\gamma})$

see that $A_1 =$  $\simeq$ , hence $\pi_1(A_1) \cong \mathbb{Z} * \mathbb{Z}$
$= \langle \alpha, \gamma \rangle$

and $A_2 = \mathbb{R}^3$, hence $\pi_1(A_2) = 0$.

<u>So now</u>: $i_{12} : \pi_1(A_1 \cap A_2) \longrightarrow \pi_1(A_1)$

 $\longmapsto$ 

$i_{12}(\alpha) = a$

$i_{12}(\gamma\beta\bar{\gamma}) = bab^{-1}$

$\langle \alpha, \gamma\beta\bar{\gamma} \rangle \qquad \langle \alpha, \gamma \rangle$

$i_{21} : \pi_1(A_1 \cap A_2) \longrightarrow \pi_1(A_2) = 1$,

hence $i_{21}(\alpha) = 1$

$i_{21}(\gamma\beta\bar{\gamma}) = 1$

Then $\pi_1(X) = \pi_1(A_1) * \pi_1(A_2) \big/ \langle i_{12}(\alpha)\, i_{21}(\alpha)^{-1},\ i_{12}(\gamma\beta\bar{\gamma})\, i_{21}(\gamma\beta\bar{\gamma})^{-1} \rangle$

$= (\underset{(a)}{\mathbb{Z}} * \underset{(b)}{\mathbb{Z}}) * 1 \big/ \langle a, bab^{-1} \rangle$

$= \langle a, b : a = 1, bab^{-1} = 1 \rangle$

$= \langle b \rangle = \mathbb{Z}$

3\. Consider the determinant function $det : M_n(\mathbb{R}) \to \mathbb{R}$ 3.
Compute the tangent map of det at I_n

We want to find $T_I det : T_I M_n(\mathbb{R}) \longrightarrow T_1 \mathbb{R}$

$$\Rightarrow T_I det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$$

Consider $A \in T_I M_n(\mathbb{R})$, i.e. $\exists \alpha : (-\varepsilon, \varepsilon) \to M_n(\mathbb{R})$ s.t. $\alpha(0) = I_n$, $\alpha'(0) = A$.

Now let $p_t(x)$ = min poly of $\alpha(t) = det(\alpha(t) - x I_n)$

$$= (x - \lambda_1(t)) \cdots (x - \lambda_n(t))$$

where $\lambda_i(t)$ are the eigenvalues of $\alpha(t)$; by their definition we can see they are continuous, and $\lambda_i(0) = 1$, $\lambda_i'(0) = \mu_i$, where μ_i e.values of A

Now: $T_I det(A) = (det \circ \alpha)'(0)$

So see that $(det \circ \alpha)(t) = det(\alpha(t)) = \prod_{i=1}^{n} \lambda_i(t)$

and then:

$$(det \circ \alpha)' = \frac{d}{dt}\left(\prod_{i=1}^{n} \lambda_i\right) = \lambda_1'(\lambda_2 \cdots \lambda_n) + \lambda_1(\lambda_2 \cdots \lambda_n)'$$

$$= \lambda_1'(\lambda_2 \cdots \lambda_n) + \lambda_1 \lambda_2'(\lambda_3 \cdots \lambda_n) + \lambda_1 \lambda_2 (\lambda_3 \cdots \lambda_n)'$$

$$= \lambda_1'(\lambda_2 \cdots \lambda_n) + \lambda_2'(\lambda_1 \lambda_3 \cdots \lambda_n) + \lambda_3'(\lambda_1 \lambda_2 \lambda_4 \cdots \lambda_n) + \lambda_1 \lambda_2 \lambda_3 (\lambda_4 \cdots \lambda_n)'$$

$$= \sum_{i=1}^{n} \lambda_i' \left(\prod_{j \neq i} \lambda_j\right)$$

Hence $T_I det(A) = (det \circ \alpha)'(0) = \sum_{i=1}^{n} \lambda_i'(0) \underbrace{\left(\prod_{j \neq i} \lambda_j(0)\right)}_{=1}$ since $\lambda_j(0) = 1$ for all j

$$= \sum_{i=1}^{n} \mu_i = tr(A)$$

(4.) M cpt, orntble, $\dim M = n$, $\partial M \approx S^{n-1}$ via $f: \partial M \to S^{n-1}$.
Let $F: M \to \mathbb{R}^n$ continuous s.t. $F|_{\partial M} = f$
Show that $F(M)$ contains $0 \in \mathbb{R}^n$ (the center of S^{n-1})

Suppose that $0 \notin F(M)$. Then $F(M) \subseteq \mathbb{R}^n \setminus \{0\}$ and we have:

$$\begin{array}{ccccccccc} \partial M & \xhookrightarrow{i} & M & \xrightarrow{F} & \mathbb{R}^n \setminus \{0\} & \xrightarrow{r} & S^{n-1} & \xrightarrow{f^{-1}} & \partial M \\ p & \longmapsto & p & \longmapsto & f(p) \in S^{n-1} & \longmapsto & f(p) & \longmapsto & p \end{array}$$

where r is the deformation retraction $r(x) = \frac{x}{\|x\|}$

Hence $f^{-1} \circ r \circ F \circ i = id_{\partial M}$; now consider the induced homomorphisms on homology:

$$\begin{array}{ccccc} H_{n-1}(\partial M) & \xrightarrow[i^*]{} & H_{n-1}(M) & \xrightarrow[(f^{-1} \circ r \circ F)^*]{} & H_{n-1}(\partial M) \\ \wr\wr & & & & \wr\wr \\ H_{n-1}(S^{n-1}) \cong \mathbb{Z}\{\alpha\} & & & & H_{n-1}(S^{n-1}) \cong \mathbb{Z}\{\alpha\} \end{array}$$

Now, see that $H_{n-1}(\partial M)$ is generated by a single $(n-1)$-cell that is the boundary of M, hence $i(\alpha)$ boundary in M, i.e. $i^*(\alpha) = 0$ in $H_{n-1}(M)$. Then:

$(f^{-1} \circ r \circ F \circ i)^*(\alpha) = (f^{-1} \circ r \circ F)^* \circ (i)^*(\alpha) = (f^{-1} \circ r \circ F)^*(0) = 0$, a contradiction since we showed $(f^{-1} \circ r \circ F \circ i)^* = id^*$

Therefore, $0 \in F(M)$

(5.) $\Omega = \{(x,y) : 1 < x^2+y^2 < 10\} \subseteq \mathbb{R}^2$, $\omega = \frac{x\,dy - y\,dx}{4x^2+y^2}$

(a) Show ω is closed in Ω

(b) Show ω is not exact in Ω.

(a) $$d(\omega) = d\left(\frac{x}{4x^2+y^2}\right)\wedge dy - d\left(\frac{y}{4x^2+y^2}\right)\wedge dx$$

Now see:

$$d\left(\frac{x}{4x^2+y^2}\right) = \frac{\partial}{\partial x}\left(\frac{x}{4x^2+y^2}\right)dx + \frac{\partial}{\partial y}\left(\frac{x}{4x^2+y^2}\right)dy$$
$$= \left(\frac{y^2-4x^2}{(4x^2+y^2)^2}\right)dx + \left(\frac{-2xy}{(4x^2+y^2)^2}\right)dy$$

and:

$$d\left(\frac{y}{4x^2+y^2}\right) = \frac{\partial}{\partial x}\left(\frac{y}{4x^2+y^2}\right)dx + \frac{\partial}{\partial y}\left(\frac{y}{4x^2+y^2}\right)dy$$
$$= \left(\frac{-8xy}{(4x^2+y^2)^2}\right)dx + \left(\frac{4x^2-y^2}{(4x^2+y^2)^2}\right)dy$$

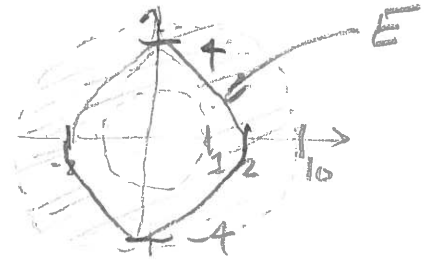
So then

$$d(\omega) = \left[\frac{y^2-4x^2}{(4x^2+y^2)^2}dx + \frac{-2xy}{(4x^2+y^2)^2}dy\right]\wedge dy$$
$$- \left[\frac{-8xy}{(4x^2+y^2)^2}dx + \frac{4x^2-y^2}{(4x^2+y^2)^2}dy\right]\wedge dx$$
$$= \frac{y^2-4x^2}{(4x^2+y^2)^2}dx\wedge dy - \frac{4x^2-y^2}{(4x^2+y^2)^2}dy\wedge dx$$

(since $dx\wedge dx = -dx\wedge dx \Rightarrow dx\wedge dx = 0$)

$$= \frac{y^2-4x^2}{(4x^2+y^2)^2}dx\wedge dy + \frac{4x^2-y^2}{(4x^2+y^2)^2}dx\wedge dy = 0$$

hence $d\omega = 0$, i.e. ω closed.

(b) See that: $\Omega =$ 

Let E be the ellipse $4x^2+y^2=16$, i.e. $E \subseteq \Omega$

Recall that the E is parametrized by:

$x(t) = 2\cos t$, $y(t) = 4\sin t$, $0 \le t \le 2\pi$

Now integrate:

$$\int_E \omega = \int_{\partial E} \frac{x\,dy - y\,dx}{4x^2+y^2}$$

$$= \frac{1}{16}\int_{\partial E} x\,dy - y\,dx$$

$$= \frac{1}{16}\int_0^{2\pi} \left(x(t)\,y'(t) - y(t)\,x'(t)\right) dt$$

$$= \frac{1}{16}\int_0^{2\pi} \left((2\cos t)(4\cos t) - (4\sin t)(-2\sin t)\right) dt$$

$$= \frac{1}{16}\int_0^{2\pi} 8(\cos^2 t + \sin^2 t)\,dt$$

$$= \frac{1}{2}\int_0^{2\pi} dt = \frac{2\pi}{2} = \pi \neq 0,$$

But, if ω were exact, by Stokes we get:

$$\int_E \omega = \int_E d\alpha = \int_{\partial \tilde{E}} d\alpha = \int_{\tilde{E}} d^2\alpha = \int_{\tilde{E}} 0 = 0$$

(where $\tilde{E}$ is ellipsoid bdd by E), which is a contradiction

Hence ω is not exact

(6.) Consider $\varphi : \mathbb{R}^2 \to \mathbb{RP}^2$ with $\varphi(x,y) = [x,y,1]$ 5.

If $C = \{(x,y) \in \mathbb{R}^2 : y^2 = x^3 - x\}$, show that $\overline{\varphi(C)}$ is a differentiable submanifold of $\mathbb{RP}^2$.

See that $\varphi(C) = \{[x,y,1] : y^2 = x^3 - x\}$

$= \{[\frac{x}{z}, \frac{y}{z}, 1] : (\frac{y}{z})^2 = (\frac{x}{z})^3 - (\frac{x}{z}),\ z \neq 0\}$

$= \{[x,y,z] : zy^2 = x^3 - z^2x,\ z \neq 0\}$

(recall $[0,0,0] \notin \mathbb{RP}^2$)

Now, if $z=0$, then $x^3 = 0$, hence $z \neq 0$ only excludes the point $[0,1,0]$, i.e. $\varphi(C) \cup \{[0,1,0]\} = \{[x,y,z] : zy^2 = x^3 - z^2x\}$

Now, note that $\varphi(C) \subseteq \varphi(C) \cup \{[0,1,0]\} = \{[x,y,z] : zy^2 = x^3 - z^2x\}$, but this set is closed since for $f : \mathbb{RP}^2 \to \mathbb{R}$, $[x,y,z] \mapsto zy^2 + z^2x - x^3$ we have

$f^{-1}(0) = \varphi(C) \cup \{[0,1,0]\}$, hence $\varphi(C) \cup \{[0,1,0]\}$ is closed since it is the inverse image of a closed set ($\{0\}$) in $\mathbb{R}$ under continuous map f. Therefore $\varphi(C) \cup \{[0,1,0]\}$ is the smallest closed set containing $\varphi(C)$, hence $\overline{\varphi(C)} = \varphi(C) \cup \{[0,1,0]\} = \{[x,y,z] : zy^2 = x^3 - z^2x\}$

Now we may proceed by the Regular value theorem, i.e. since $\overline{\varphi(C)} = f^{-1}(0)$, we must show T_pf surj. $\forall p \in f^{-1}(0)$:

$$T_pf : T_p\mathbb{RP}^2 \to T_p\mathbb{R}$$

$$T_pf = [z^2 - 3x^2,\ 2zy,\ 2zx + y^2]\Big|_{p=[x,y,z]}.$$

Since $p \in \mathbb{RP}^2$, all of x,y,z cannot be zero, and since codom$(T_pf) = \mathbb{R}$ and linear, must only show one elt nonzero and get surjectivity.

- $\underline{x \neq 0} \Rightarrow z^2 - 3x^2 \neq 0$ (done) $\underline{\text{or}}$ $z^2 \neq 0 \Rightarrow z \neq 0 \Rightarrow 2zx + y^2 \neq 0$ (done) or $y^2 \neq 0 \Rightarrow y \neq 0 \Rightarrow 2zy \neq 0$ (done).
- $\underline{y \neq 0} \Rightarrow \underset{\text{(done)}}{2zy \neq 0}$ or $z = 0 \Rightarrow 2zx + y^2 = y^2 \neq 0$ (done) $\Rightarrow T_pf$ surjective
- $\underline{z \neq 0} \Rightarrow z^2 - 3x^2 \neq 0$ (done) or $x \neq 0 \Rightarrow 2zx + y^2 \neq 0$ (done) or $y^2 \neq 0 \Rightarrow 2zy \neq 0$ (done)

(7) M, N cpt, dim n, $f: M \to N$ with $\deg f \neq 0$
Show $f_*(\pi_1(M, x_0)) \leq \pi_1(N, f(x_0))$ has finite index :

N cpt, connected mfld $\Rightarrow$ conn, path conn, semiloc s-conn.
$\Rightarrow$ apply classification of connected covers.

Let $p: \tilde{N} \to N$ be a cover corresponding to $f_*(\pi_1(M, x_0)) \leq \pi_1(N, f(x_0))$
Then $f_*(\pi_1(M, x_0)) = p_*(\pi_1(\tilde{N}, \tilde{x}_0))$ by construction, hence by the lifting criterion $\exists \tilde{f}$ such that $f = p \circ \tilde{f}$ $\rightarrow$ $\left[\begin{tikzcd} & \tilde{N} \arrow[d, "p"] \\ M \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & N \end{tikzcd}\right]$

So then by construction:

$$\#\text{sheets of } p = [p_*(\pi_1(\tilde{N}, \tilde{x}_0)) : \pi_1(N, f(x_0))]$$
$$= [f_*(\pi_1(M, x_0)) : \pi_1(N, f(x_0))]$$

Hence we're left to show that p finite-sheeted cover. (i.e. $\deg p < \infty$)

Suppose not ; i.e. p an infinite sheeted covering space, hence $\tilde{N}$ is not compact.

Recall $\tilde{f}: M \to \tilde{N}$, M compact, $\tilde{N}$ non-compact $\Rightarrow \deg \tilde{f} = 0$

This a contradiction since
$0 \neq \deg f = \deg p \circ \tilde{f} = \deg p \cdot \deg \tilde{f}$,
therefore $\tilde{N}$ is compact; a compact cover of a compact space is finite-sheeted, hence p is finite-sheeted, hence

If $\tilde{f}$ surjective, then $\tilde{N} = \tilde{f}(M)$ is compact, a contradiction. So $\tilde{f}$ not surjective, hence $\deg \tilde{f} = 0$

$$[f_*(\pi_1(M, x_0)) : \pi_1(N, f(x_0))] = [p_*(\pi_1(\tilde{N}, \tilde{x}_0)) : \pi_1(N, f(x_0))] < \infty$$