2007, Fall

Problem 1.

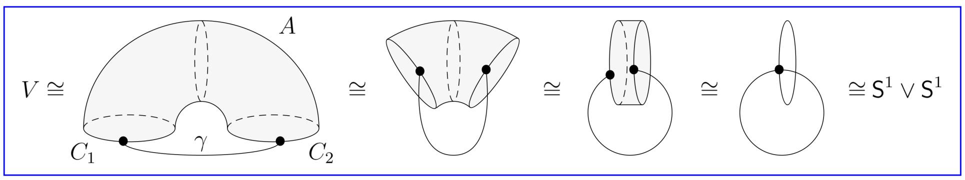
We already know that $\mathsf{H}_0(X \times \mathsf{S}^n) \cong 0$ by path connectedness. If $j \geq 1$, then by Künneth,

$$\mathsf{H}_j(X \times \mathsf{S}^n) \cong [\underbrace{\mathsf{H}_0(X)}_{\cong \mathbb{Z}} \otimes \mathsf{H}_j(\mathsf{S}^n)] \oplus [\mathsf{H}_j(X) \otimes \underbrace{\mathsf{H}_0(\mathsf{S}^n)}_{\cong \mathbb{Z}}] \oplus \bigoplus_{\substack{k,\ell \geq 1 \\ k+\ell=j}} \underbrace{\mathsf{H}_k(X)}_{\cong 0} \otimes \mathsf{H}_\ell(\mathsf{S}^n) \cong \mathsf{H}_j(X) \oplus \mathsf{H}_j(\mathsf{S}^n).$$

Thus $\mathsf{H}_j(X \times \mathsf{S}^n) \cong \begin{cases} \mathbb{Z} & j = 0, n, \\ 0 & \text{else.} \end{cases}$ □

Problem 2.

Let $U \cong \mathbb{R}^3$, and let V be the union of the attached handle and a curve γ connecting C_1 to C_2. Then U is equivalent to a wedge of two circles, and $U \cap V$ is the "pair of handcuffs" $C_1 \cup \gamma \cup C_2$.



Denote by $i : U \cap V \hookrightarrow U$ and $j : U \cap V \hookrightarrow V$ the canonical inclusions. Then $i_* : \pi_1(U \cap V) \to \pi_1(U)$ is trivial since $\pi_1(\mathbb{R}^3) \cong 1$, so it remains to determine $j_* : \pi_1(U \cap V) \to \pi_1(V)$. Thinking of C_1, C_2, and γ as oriented paths, we see that $\pi_1(U \cap V)$ is generated by the loops $[C_1]$ and $[\gamma * C_2 * \gamma^{-1}]$, and $\pi_1(V)$ is generated by $[C_1]$ and $[\gamma]$. The inclusion j sends $[C_1]$ to itself, but identifies $[C_1]$ and $[C_2]$, which are now connected by the 2-cell A. Hence $j_*([C_1]) = [C_1]$ and $j_*([\gamma * C_2 * \gamma^{-1}]) = [\gamma * C_1 * \gamma^{-1}]$. So by van Kampen

$$\pi_1(X) \cong \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) \cong \frac{1 * (\mathbb{Z}\langle [C_1] \rangle * \mathbb{Z}\langle [\gamma] \rangle)}{\langle i_*([C_1]) j_*([C_1])^{-1}, i_*([\gamma * C_2 * \gamma^{-1}]) j_*([\gamma * C_1 * \gamma^{-1}])^{-1} \rangle}$$

$$\cong \frac{\langle [C_1], [\gamma] \rangle}{\langle [C_1]^{-1}, [\gamma * C_2 * \gamma^{-1}]^{-1} \rangle} \cong \langle x, y \mid x = yxy^{-1} = 1 \rangle \cong \langle y \mid yy^{-1} = 1 \rangle \cong \mathbb{Z}.$$

□

Problem 3.

We may think of det as a function $\mathbb{R}^{n^2} \to \mathbb{R}$, with $\mathbb{R}^{n^2}$ being coordinatized by $(x_{ij})_{1 \leq i,j \leq n}$. Then for the matrix $I = (x_{ij})_{1 \leq i,j \leq n} \in \mathsf{Mat}_n(\mathbb{R})$, denoting by $I_{ij} \in \mathsf{Mat}_{n-1}(\mathbb{R})$ the matrix obtained from I by deleting the i-th row and j-th column, for $1 \leq i, j \leq n$, we have

$$\mathsf{det}(I) = \sum_{1 \leq i,j \leq n} x_{ij} (-1)^{i+j} \mathsf{det}(I_{ij}) \implies \left(\frac{\partial}{\partial x_{ij}} \mathsf{det} \right)(I) = (-1)^{i+j} \mathsf{det}(I_{ij}) = \begin{cases} 0 & i \neq j, \\ 1 & i = j. \end{cases}$$

Hence for any matrix $v = (v)_{1 \leq i,j \leq n} \in \mathsf{T}_I \mathsf{Mat}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$, we have

$$(\mathsf{d}(\mathsf{det}))_I(v) = \sum_{1 \leq i,j \leq n} \left(\frac{\partial}{\partial x_{ij}} \mathsf{det} \right)(I) v_{ij} = \sum_{j=1}^{n} v_{jj} = \mathsf{tr}(v),$$

where tr denotes the trace. Hence $(\mathsf{d}(\mathsf{det}))_I = \mathsf{tr}$ as maps from $\mathsf{T}_I \mathsf{Mat}_n(\mathbb{R}) \cong \mathsf{Mat}_n(\mathbb{R})$ to $\mathbb{R}$. □

Problem 4.

The sphere S^{n-1} is a deformation retract of $\mathbb{R}^n \setminus 0$ via the normalization map $u : \mathbb{R}^n \setminus 0 \twoheadrightarrow \mathsf{S}^{n-1}$ given by $u(x) := x/\|x\|$, so assuming that $0 \notin \mathrm{im}(F)$, we have a well defined composite $f^{-1} \circ u \circ F : M \to \partial M$ which fits into the diagram

$$\partial M \xhookrightarrow{\ \iota\ } M \xrightarrow{\ F\ } \mathbb{R}^n \setminus 0 \overset{u}{\twoheadrightarrow} \mathsf{S}^{n-1} \xrightarrow{\ f^{-1}\ } \partial M,$$

$$\mathsf{H}_{n-1}(\partial M) \xrightarrow{\ \iota_*\ } \mathsf{H}_{n-1}(M) \xrightarrow{\ (f^{-1}\circ u\circ F)_*\ } \mathsf{H}_{n-1}(\partial M).$$

Notice that the composite function $\partial M \to \partial M$ along the top row is $\mathrm{id}_{\partial M}$ since $F\big|_{\partial M} = f$, so the induced composite function $\mathsf{H}_{n-1}(\partial M) \to \mathsf{H}_{n-1}(\partial M)$ along the bottom row is certainly nonzero. However, the single generator $[\partial M] \in \mathsf{H}_{n-1}(\partial M)$ is clearly mapped to a boundary in $\mathsf{H}_{n-1}(M)$ by ι_*, and so

$$(f^{-1} \circ u \circ F)_* \circ \iota_*([\partial M]) = (f^{-1} \circ u \circ F)_*(0) = 0,$$

a contradiction. □

Problem 5.

(a) We have that

$$\mathsf{d}\omega = \mathsf{d}\left(\frac{x}{4x^2+y^2}\right) \wedge \mathsf{d}y - \mathsf{d}\left(\frac{y}{4x^2+y^2}\right) \wedge \mathsf{d}x = \left[\frac{\partial}{\partial x}\left(\frac{x}{4x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{4x^2+y^2}\right)\right] \mathsf{d}x \wedge \mathsf{d}y$$
$$= \left[\frac{-4x^2+y^2}{(4x^2+y^2)^2} + \frac{4x^2-y^2}{(4x^2+y^2)^2}\right] \mathsf{d}x \wedge \mathsf{d}y = 0.$$

(Note that the denominators above are never 0 on Ω.) □

(b) Consider the ellipse $X \subset \Omega$ defined by the equation $4x^2 + y^2 = 4^2$. If $\omega = \mathsf{d}\eta$ for some $\eta \in \Omega^0(\Omega)$, then by Stokes $\int_X \omega = \int_{\partial X} \eta = 0$ since $\partial X = \varnothing$. But parametrizing X by $x(t) := 2\mathsf{cos}(t)$ and $y(t) := 4\mathsf{sin}(t)$ for $0 \le t < 2\pi$, we have

$$\int_X \omega = \int_0^{2\pi} \frac{x(t)y'(t) - y(t)x'(t)}{4^2} \mathsf{d}t = \frac{1}{16}\int_0^{2\pi} 8\left[\mathsf{cos}^2(t) + \mathsf{sin}^2(t)\right] \mathsf{d}t = \pi,$$

so ω can't be exact on Ω. □

Problem 6.

- The set

$$\varphi(C) = \{[x:y:1] \in \mathbb{R}\mathsf{P}^2 \mid y^2 - x^3 + x = 0\} = \left\{\left[\frac{x}{z} : \frac{y}{z} : 1\right] \,\middle|\, \frac{y^2}{z^2} - \frac{x^3}{z^3} + \frac{x}{z} = 0, z \in \mathbb{R} \setminus 0\right\}$$
$$= \{[x:y:z] \mid y^2 z - x^3 + xz^2 = 0, z \in \mathbb{R} \setminus 0\}.$$

isn't closed since it doesn't include the case $z = 0$. Consider the equation $y^2 z - x^3 + xz^2 = 0$ with $z = 0$; regardless of the choice of $y \in \mathbb{R}$, this equation yields $x = 0$, so the only element of $\mathbb{R}\mathsf{P}^2$ satisfying $y^2 z - x^3 + xz^2 = 0$ which doesn't already belong to $\varphi(C)$ is $[0:1:0]$. Then defining $f : \mathbb{R}\mathsf{P}^2 \to \mathbb{R}$ by $f([x:y:z]) := y^2 z - x^3 + xz^2$, we have that

$$f^{-1}(0) = \{[x:y:z] \mid y^2 z - x^3 + xz^2 = 0\} = \varphi(C) \cup \{[0:1:0]\}.$$

This set is closed since it's the preimage of the closed point $\{0\} \subset \mathbb{R}$ under the continuous map f, and is also obviously the smallest closed set containing $\varphi(C)$. Thus $f^{-1}(0) = \overline{\varphi(C)}$.

- So to show that $\overline{\varphi(C)}$ is a submanifold of $\mathbb{R}\mathsf{P}^2$, we need only verify that 0 is a regular value of f. To see this, let $[x : y : z] \in f^{-1}(0)$ and consider the map $\mathsf{d}f_{[x:y:z]} : \mathsf{T}_{[x:y:z]}\mathbb{R}\mathsf{P}^2 \to \mathsf{T}_0\mathbb{R}$,

$$\mathsf{d}f_{[x:y:z]} = \begin{pmatrix} -3x^2 + z^2 & 2yz & y^2 + 2xz \end{pmatrix}.$$

  This linear map is surjective as long as one of its entries is nonzero. Now, at least one of x, y, z is nonzero by definition of $\mathbb{R}\mathsf{P}^2$, so we have the following cases.

  - Suppose $x \neq 0$. If $z = 0$ then $-3x^2 + z^2 \neq 0$. If $z \neq 0$ and $y = 0$ then $y^2 + 2xz \neq 0$. If $z \neq 0$ and $y \neq 0$ then $2yz \neq 0$.
  - Suppose $y \neq 0$. If $z \neq 0$ then $2yz \neq 0$. If $z = 0$ then $y^2 + 2xz \neq 0$.
  - Suppose $z \neq 0$. If $y \neq 0$ then $2yz \neq 0$. If $y = 0$ and $x \neq 0$ then $y^2 + 2xz \neq 0$. If $y = x = 0$ then $-3x^2 + z^2 \neq 0$.

  Hence 0 is indeed a regular value of f.

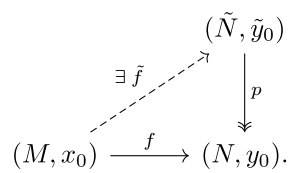
□

Problem 7.

Let $y_0 := f(x_0) \in N$, and let $p : (\tilde{N}, \tilde{y}_0) \twoheadrightarrow (N, y_0)$ be the covering space corresponding to the subgroup $f_*(\pi_1(M, x_0)) \subset \pi_1(N, y_0)$. Then

$$k := [\pi_1(N, y_0) : p_*(\pi_1(\tilde{N}, \tilde{y}_0))] = [\pi_1(N, y_0) : f_*(\pi_1(M, x_0))],$$

and p is a k-sheeted covering of (M, x_0). So we're done if we can show that $k < \infty$. Assume that $k = \infty$. Now by definition of p, there exists a lift



Since M is compact, then so is $\mathsf{im}(\tilde{f}) \subset \tilde{N}$. But $\tilde{N}$ is certainly noncompact since it's an ∞-sheeted covering space, and so $\tilde{f}$ is nonsurjective. Then $\mathsf{deg}(\tilde{f}) = 0$, and thus $\mathsf{deg}(f) = \mathsf{deg}(p)\mathsf{deg}(\tilde{f}) = 0$. But this is a contradiction since $\mathsf{H}_n(f) : \mathsf{H}_n(M) \to \mathsf{H}_n(N)$ is nontrivial by assumption. □