1. If $f$ were not surjective, $\deg f = 0$.

2. Van Kampen

3. Lifting criterion and Covering spaces

4. Long Exact Sequence for relative homology

5. Use pullbacks and $H^n_{dR}(M) = 0$

6. Regular Value Theorem, use cross product.

7. $SO(3)$ acts transitively on the tangent bundle of $S^2$.

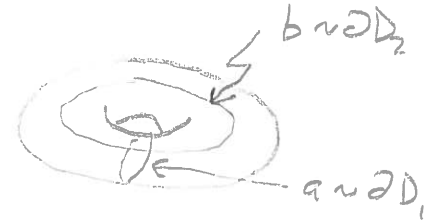
(1.) $M, N$ cpt, ornt, dim. $n$, $f: M \to N$ differentiable.
Suppose $f^*: H^n_{dR}(N) \to H^n_{dR}(M)$ is surjective. Show f
Then prove $f$ is surjective:

Suppose that $f$ is not surjective. Then $\exists\, y \in N$ such that $y \notin f(M)$, hence the fiber $f^{-1}(y)$ ~~is~~ is empty, hence the geom. degree of $f$ is zero here, hence $\deg f = 0$. Since $y$ is trivially a regular value and $\deg f$ must agree with geometric degree at all regular values.

Recall that since $M, N$ compact we have top-dim de Rham isomorphisms $I: H^n(M) \to \mathbb{R}$, $I: H^n(N) \to \mathbb{R}$, and then the map $I \circ f^* \circ I^{-1} \equiv 0$ since $\deg f = 0$, hence $I \circ f^* \circ I^{-1}$ not surjective, hence $f^*$ not surjective since we already know the $I$ maps are isomorphisms.

This is a contradiction since we assumed $f^*$ was surjective; hence $f$ is surjective.

(2.) $D^2 \subseteq \mathbb{C}$ closed unit disk.
Consider torus $T^2 = S^1 \times S^1$ and two copies $D_1, D_2$ of $D^2$.
For integers $p, q$, let $X_{p,q} = T^2 \amalg D_1 \amalg D_2 /\sim$
where $e^{i\theta} \in \partial D_1 \sim (e^{ip\theta}, 1) \in S^1_a \times S^1_b$
$e^{i\theta} \in \partial D_2 \sim (1, e^{iq\theta}) \in S^1_a \times S^1_b$




Compute $X_{p,q}$:

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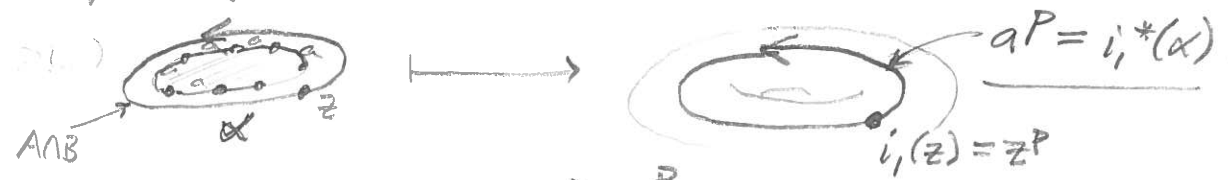
We'll apply Seifert–van Kampen twice.
Step 1: Let $X_p := T^2 \amalg D_1 /\sim$ where $e^{i\theta} \in \partial D_1 \sim (e^{ip\theta}, 1) \in T^2$
Then let $A \simeq T^2$, $B = D_1$, $A \cap B = \partial D_1 \simeq S^1$, $A \cup B = X_p$:


See that $\pi_1(A) \cong \pi_1(T^2) \cong \mathbb{Z}\{a\} \oplus \mathbb{Z}\{b\}$ (see fig.)
and: $\pi_1(A \cap B) \cong \pi_1(S^1) \cong \mathbb{Z}\{\alpha\}$.
Now consider the inclusion maps:

$$i_1^*: \pi_1(A \cap B) \cong \langle \alpha \rangle \longrightarrow \pi_1(A) \cong \langle a \rangle \oplus \langle b \rangle.$$

$\partial D_1$




$$\alpha \longmapsto a^p$$

and $i_2^*: \pi_1(A \cap B) \longrightarrow \pi_1(B) \cong \pi_1(D^2) \cong 1$, hence $i_2^*(\alpha) = 1$.

So by Seifert–van Kampen:

$$\pi_1(X_p) = \pi_1(A) * \pi_1(B) \Big/ \langle i_1^*(\alpha) i_2^*(\alpha)^{-1} \rangle = (\mathbb{Z}\{a\} \oplus \mathbb{Z}\{b\}) * 1 \Big/ \langle a^p \rangle$$

$$\cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}$$

2 cont'd :



Step 2: Now see that $X_{p,q} = T^2 \amalg D_1 \amalg D_2 / \sim \; = X_p \amalg D_2 / \sim$

Let $A = X_p$, $B = D_2$, $A \cap B = \partial D_2 \simeq S^1$.

Now see that $\pi_1(X_p) = \mathbb{Z}\{a\}/\langle a^p \rangle \oplus \mathbb{Z}\{b\}$

and $\pi_1(A \cap B) \cong \pi_1(\partial D_2) \cong \pi_1(S^1) \cong \mathbb{Z}\{\beta\}$

and consider the inclusions:

$$i_1^*: \pi_1(A \cap B) \cong \langle \beta \rangle \longrightarrow \pi_1(A) \cong \langle a : a^p \rangle \oplus \langle b \rangle$$

$\partial D_2$:


$\underline{i_1^*(\beta) = b^q}$

$$\beta \longmapsto b^q$$

and $i_2^*: \pi_1(A \cap B) \longrightarrow \pi_1(B) \cong 1$, hence $\underline{i_2^*(\beta) = 1}$.

So by Seifert-van Kampen:

$$\pi_1(X_{p,q}) = \pi_1(A) * \pi_1(B) / \langle i_1^*(\beta) i_2^*(\beta)^{-1} \rangle$$

$$\cong \left( \mathbb{Z}\{a\}/\langle a^p \rangle \oplus \mathbb{Z}\{b\} \right) * 1 \Big/ \langle b^q \rangle$$

$$\cong \mathbb{Z}\{a\}/\langle a^p \rangle \oplus \mathbb{Z}\{b\}/\langle b^q \rangle \cong \underline{\underline{\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/q\mathbb{Z}}}$$

(3.) Any two continuous maps $f, g : X \to S^1$ where $\pi_1(X) = 1$ are homotopic :

Consider the covering map $p : \mathbb{R} \to S^1$.
Since $\pi_1(X)$ is trivial, we get that $f_*, g_* : \pi_1(X) \to \pi_1(S^1)$ are both trivial, hence

$$f_*(\pi_1(X)) \subseteq p_*(\pi_1(\mathbb{R}))$$
$$g_*(\pi_1(X)) \subseteq p_*(\pi_1(\mathbb{R})),$$

i.e. the lifting criterion is satisfied, hence we obtain:

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & S^1 \end{tikzcd} \quad \& \quad \begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ X \arrow[ur, "\tilde{g}"] \arrow[r, "g"'] & S^1 \end{tikzcd}$$

Now, $\mathbb{R}$ is contractible, hence $\exists$ homotopy $h_t : \mathbb{R} \to \mathbb{R}$ such that $h_0 = id_{\mathbb{R}}$ and $h_1 = const$; then see that $\tilde{f} \circ h_0 = \tilde{f}$, $\tilde{g} \circ h_0 = \tilde{g}$, and $\tilde{f} \circ h_1 = const.$, $\tilde{g} \circ h_1 = const.$, therefore $\tilde{f}_t = \tilde{f} \circ h_t$ and $\tilde{g}_t = \tilde{g} \circ h_t$ give homotopies from $\tilde{f}, \tilde{g}$ to const., i.e. $\tilde{f}$ and $\tilde{g}$ are homotopic.

Thus let $F_t$ be the homotopy such that $F_0 = \tilde{f}$ and $F_1 = \tilde{g}$ and consider the new homotopy $G_t := p \circ F_t$. Then:

$$G_0 = p \circ F_0 = p \circ \tilde{f} = f$$
$$G_1 = p \circ F_1 = p \circ \tilde{g} = g \quad , \quad \text{hence } f \simeq g.$$

(4.) Calculate the relative homology groups $H_n(S^1 \times D^2, S^1 \times \partial D^2)$ where $D^2$ = closed disk, $S^1$ = circle.

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Recall the long exact sequence for relative homology:

$$\cdots \longrightarrow H_n(A) \xrightarrow[i_*]{} H_n(X) \xrightarrow[j_*]{} H_n(X,A) \xrightarrow[\partial]{} H_{n-1}(A) \to H_{n-1}(X) \to \cdots \to H_0(X,A) \to 0$$

Let $X = S^1 \times D^2$ = solid torus $\simeq S^1$

$A = S^1 \times \partial D^2 \cong S^1 \times S^1 \simeq T^2$ torus.

Then $H_n(X) \cong H_n(S^1) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z} & n=1 \\ 0 & n>1 \end{cases}$, $H_n(A) \cong H_n(T^2) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}\oplus\mathbb{Z} & n=1 \\ \mathbb{Z} & n=2 \\ 0 & n>2 \end{cases}$

and then we get the sequence:

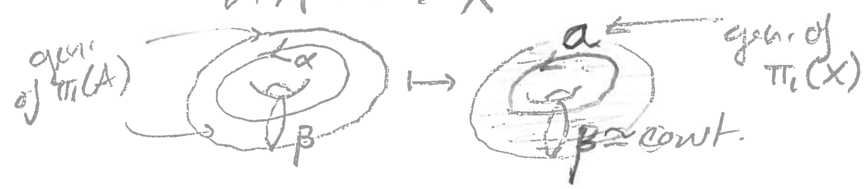
$$\underbrace{0 \to H_3(X,A) \to \mathbb{Z} \to 0}_{} \longrightarrow H_2(X,A) \to \mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z} \to H_1(X,A) \to \mathbb{Z} \to \mathbb{Z} \to H_0(X,A) \to 0$$

↳ By exactness, $\underline{H_3(X,A) \cong \mathbb{Z}}$

- Consider the part $0 \to H_2(X,A) \to H_1(A) \xrightarrow{i_*} H_1(X)$ and the inclusion:

$H_1(A) \cong \langle\alpha\rangle \oplus \langle\beta\rangle$, $H_1(X) \cong \langle a \rangle$

$i: A \hookrightarrow X$




Then clearly the generator $\alpha$ is sent to generator $a$ and the generator $\beta$ is sent to a contractible loop, hence $i_*(\alpha) = a$ and $i_*(\beta) = 0$, i.e. $\ker i_* = \mathbb{Z}$ & $\operatorname{im} i_* = \mathbb{Z}$.

By exactness, $H_2(X,A) \xrightarrow{\partial} H_1(A)$ is injective, hence $H_2(X,A) \cong \ker i_* = \mathbb{Z}$. $\Rightarrow$ $\underline{H_2(X,A) \cong \mathbb{Z}}$

- Now consider the part $H_1(X) \xrightarrow{j_*} H_1(X,A) \xrightarrow{\partial} H_0(A) \xrightarrow{i_*} H_0(X) \longrightarrow$

$A \subseteq X$ and both path connected, hence this $i_*$ is injective, i.e. $\underline{0 = \ker i_* = \operatorname{im} \partial}$; but we saw $\mathbb{Z} = \operatorname{im}(i_*: H_1(A) \to H_1(X)) = \ker j_*$, hence $\operatorname{im} j_* = 0$ since $H_1(X) \cong \mathbb{Z}$, i.e. $\underline{0 = \operatorname{im} j_* = \ker \partial}$, hence $\underline{H_1(X,A) \cong 0}$

7 cont'd :

Lastly consider the part $\underset{\mathbb{Z}}{\overset{\cong}{H_1(A)}} \xrightarrow{i_*} \underset{\mathbb{Z}}{\overset{\cong}{H_0(X)}} \xrightarrow{j_*} H_0(X,A) \longrightarrow 0$.

$j_*$ is surjective by exactness, hence $H_0(X,A) \cong \operatorname{im} j_*$; we already showed that $i_*$ is injective, hence $\mathbb{Z} \cong \operatorname{im} i_* \cong \ker j_*$, hence $\operatorname{im} j_* \cong 0$, hence $\underline{H_0(X,A) \cong 0}$

Thus we've obtained: $H_n(X,A) = \begin{cases} \mathbb{Z} & n=3 \\ \mathbb{Z} & n=2 \\ 0 & n=1 \\ 0 & n=1 \end{cases}$

(5.) $M$ cpt, ornt, dim $n$, $H^1_{dR}(M)=0$, $f: M \to T^n$ smooth.
Show $\deg f = 0$.

$T^n = S^1 \times \cdots \times S^1$ is an orientable manifold, hence it will have a volume form. Consider the coordinate system on $T^n$ given by $\theta_i : S^1 \to \mathbb{R}$, hence we may write our volume form on $T^n$ as $d\theta_1 \wedge \cdots \wedge d\theta_n \in \Omega^n(T^n)$

Consider the pullback map $f^*: \Omega^k(T^n) \to \Omega^k(M)$; then:

$$f^*(d\theta_1 \wedge \cdots \wedge d\theta_n) = f^*(d\theta_1) \wedge \cdots \wedge f^*(d\theta_n) \in \Omega^n(M)$$

where the $d\theta_i \in \Omega^1(T^n)$; clearly the $d\theta_i$ are closed, hence $[d\theta_i] \in H^1(T^n)$ well-defined, hence $f^*(d\theta_i) \in H^1(M) = 0$, hence $[f^*(d\theta_i)] = 0$ for all $i$, and then:

$$f^*(d\theta_1 \wedge \cdots \wedge d\theta_n) = f^*(d\theta_1) \wedge \cdots \wedge f^*(d\theta_n) = 0 \in H^n(M)$$

Since the class of the volume form $[d\theta_1 \wedge \cdots \wedge d\theta_n]$ generates $H^n(T^n)$ (by top-dim'l de Rham, $T^n$ cpt),

$$\left\{ \begin{array}{l} f^*: H^n(T^n) \to H^n(M) \\ \quad [\alpha] \mapsto (\deg f)\alpha \end{array} \right.$$ is thus the zero map,

hence $\deg f = 0$

(6.) Show $M=\{A\in M_{2,3}(\mathbb{R}) : \mathrm{rk}A = 1\}$ smooth manifold

See that $A=\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3\end{bmatrix} \in M_{2,3}(\mathbb{R})$ is rank 1 if $a=(a_1,a_2,a_3)$ & $b=(b_1,b_2,b_3)$ are linearly dependent, i.e. colinear in $\mathbb{R}^3$, i.e. $a\times b = 0$, so define

$$f: M_{2,3}(\mathbb{R}) \cong \mathbb{R}^6 \longrightarrow \mathbb{R}^3$$
$$A = (a,b) \longmapsto a\times b$$

hence $M=f^{-1}(0)$; now apply regular value theorem:

For $P=(a,b)\in f^{-1}(0)$, consider $T_pf : T_p\mathbb{R}^6 \longrightarrow T_0\mathbb{R}^3$

Then choose $V=(V_1,V_2)\in T_p\mathbb{R}^6$ and representative curve $\alpha(t)=tV+P$; then $T_pf(V)=(f\circ\alpha)'(0)$:

$$(f\circ\alpha)(t) = f(tV+P) = f(tV_1+a,\, tV_2+b) = (tV_1+a)\times(tV_2+b)$$

$$\Rightarrow (f\circ\alpha)'(t) = \frac{d}{dt}\left[(tV_1+a)\times(tV_2+b)\right] = \left(\frac{d}{dt}(tV_1+a)\right)\times(tV_2+b) + (tV_1+a)\times\left(\frac{d}{dt}(tV_2+b)\right)$$
$$= (V_1)\times(tV_2+b) + (tV_1+a)\times(V_2)$$

$$\Rightarrow T_pf(V) = T_{(a,b)}f(V_1,V_2) = (f\circ\alpha)'(0) = V_1\times b + a\times V_2$$

Now recall that $P=(a,b)\in f^{-1}(0)$, hence $a\times b = 0$, hence $b=\lambda a$, so we have:

$$T_pf(V_1,V_2) = V_1\times\lambda a + (a\times V_2)$$
$$= -\lambda(a\times V_1) + (a\times V_2) = a\times(V_2-\lambda V_1)$$

Now let
$$\left.\begin{aligned} u &= (1,1,1,1+\lambda,\lambda,\lambda) \\ v &= (1,1,1,\lambda,1+\lambda,\lambda) \\ w &= (1,1,1,\lambda,\lambda,1+\lambda) \end{aligned}\right\} \in T_p\mathbb{R}^6 \Rightarrow \begin{aligned} T_pf(u) &= a\times i \\ T_pf(v) &= a\times j \\ T_pf(v) &= a\times k \end{aligned}$$

which are linearly independent in $\mathbb{R}^3$, hence $T_pf$ surjective.

(Supposing $a$ is multiple of a canonical basis vector, then replace corresponding $u,v,w$ with $x=(1,1,1,1+\lambda,1+\lambda,1+\lambda)$, i.e. $T_pf(x)=a\times(1,1,1)$ & still get basis)

(7) $SO_3 = \{A \in M_3(\mathbb{R}) : \det A = 1, AA^T = I\}$

Prove that if $\omega \in \Omega^1(S^2)$ s.t. $\phi^*(\omega) = \omega$ for all $\phi \in SO_3$, then $\omega \equiv 0$.

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$SO_3$ acts on $TS^2$ via $\phi(p,v) = (\phi(p), T_p\phi(v))$ and the action is transitive, i.e. for each $(p,v), (q,w) \in TS^2$, $\exists\, \phi \in SO_3$ such that $\phi(p,v) = (q,w)$.

Now, $\phi^*(\omega) = \omega$ for all $\phi \in SO_3$, hence for $(p,v) \in TS^2$ (and since $\omega$ 1-form):

$$\omega_p(v) = \phi^*(\omega)_p(v) = \omega_{\phi(p)}(T_p\phi(v))$$

But recall that the action is transitive, hence we can choose $\phi_{p,v} \in SO_3$ s.t. $\phi_{p,v}(p,v) = (p,0)$, and then since $\phi^*(\omega) = \omega$ $\forall \phi \in SO_3$, it is true for $\phi_{p,v}$ in particular, hence for arbitrary $(p,v) \in TS^2$,

$$\omega_p(v) = \phi_{p,v}^*(\omega)_p(v) = \omega_{\phi_{p,v}(p)}(T_p\phi_{p,v}(v))$$

$$= \omega_p(0) = 0$$

Hence $\omega_p(v) = 0$ for arbitrary $(p,v)$, hence $\omega \equiv 0$.