2006, Fall

Problem 1.

Assume f is nonsurjective; then $\mathsf{deg}(f) = 0$. The map $\int_M : \mathsf{H}^n_{\mathsf{dR}}(M) \to \mathbb{R}$ is an isomorphism and the map $f^* : \mathsf{H}^n_{\mathsf{dR}}(N) \to \mathsf{H}^n_{\mathsf{dR}}(M)$ is surjective, so $\int_M f^* : \mathsf{H}^n_{\mathsf{dR}}(N) \to \mathbb{R}$ is surjective. But by definition of degree, $\int_M f^* = \mathsf{deg}(f) \int_N = 0$, and the zero map is nonsurjective. $\square$

Problem 2.

- We first set $X_p := (\mathsf{T}^2 \coprod D_1)/\sim$, where we identify each point $e^{i\theta} \in \partial D_1, 0 \leq \theta < 2\pi$, with the point $(e^{ip\theta}, 1) \in \mathsf{T}^2$. Now let $U := D_1 \subset X_p$ and $V := \mathsf{T}^2 \subset X_p$, so that $U \cup V = X_p$ and $U \cap V = \partial D_1$. Let $i : U \cap V \hookrightarrow U$ and $j : U \cap V \hookrightarrow V$ be the canonical inclusions.

  Firstly, the induced map $j_* : \pi_1(U \cap V) \to \pi_1(U)$ is trivial since U is a contractible disc. Next, observe that $\pi_1(U \cap V) \cong \pi_1(\mathsf{S}^1) \cong \mathbb{Z}$ is generated by a single loop u, and $\pi_1(V) \cong \pi_1(\mathsf{T}^2) \cong \mathbb{Z}^{\oplus 2}$ is generated by a meridianal loop x and a lateral loop y. Say w.l.o.g. that D_1 is the disc glued onto the corresponding meridianal circle of T^2. Then the induced map $i_* : \pi_1(U \cap V) \to \pi_1(V)$ sends u to x^p, so by van Kampen

$$\pi_1(X_p) \cong \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) \cong \frac{1 * \langle x, y \rangle}{\langle i_*(u) j_*(u)^{-1} \rangle} \cong \frac{\langle x, y \rangle}{\langle x^p \rangle}.$$

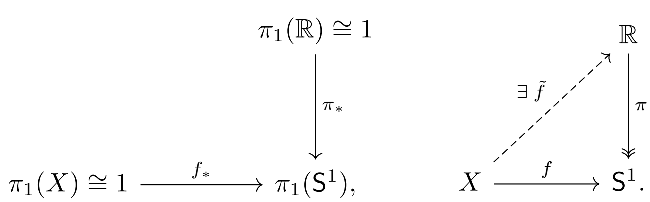
- Now observe that $X_{pq} \cong (X_p \coprod D_2)/\sim$, where we identify each point $e^{i\phi} \in \partial D_2, 0 \leq \phi < 2\pi$, with $(1, e^{iq\phi}) \in \mathsf{T}^2$. Let $R := D_2 \subset X_{pq}$ and $S := X_p \subset X_{pq}$, so that $R \cup S = X_{pq}$ and $R \cap S = \partial D_2$. Then similarly to the above,

$$\pi_1(X_{pq}) \cong \pi_1(R) *_{\pi_1(R \cap S)} \pi_1(S) \cong \frac{1 * (\langle x, y \rangle / \langle x^p \rangle)}{\langle y^q \rangle} \cong \frac{\langle x, y \rangle}{\langle x^p, y^q \rangle} \cong \mathbb{Z}_p \oplus \mathbb{Z}_q.$$

$\square$

Problem 3.

The universal cover $\pi : \mathbb{R} \twoheadrightarrow \mathsf{S}^1$ satisfies $\pi_*(\pi_1(\mathbb{R})) \supset f_*(\pi_1(X))$ since both of $\pi_1(\mathbb{R}), \pi_1(X)$ are trivial, and thus we have the lifting diagram on the right



Let $\{h_t\}_{0 \leq t \leq 1}$ be a homotopy with $h_0 = \mathsf{id}_{\mathbb{R}}$ and $h_1 = c$ for some constant map $c : \mathbb{R} \to \mathbb{R}$. Then $\{h_t \circ \tilde{f}\}_{0 \leq t \leq 1}$ gives a homotopy between $\tilde{f} : X \to \mathbb{R}$ and the constant map $c : X \to \mathbb{R}$. We likewise have a lift $\tilde{g} : X \to \mathbb{R}$, together with a homotopy between $\tilde{g}$ and c. So $\tilde{f}$ and $\tilde{g}$ are related by some homotopy $\{k_t\}_{0 \leq t \leq 1}$ with $k_0 = \tilde{f}$ and $k_1 = \tilde{g}$, and then $\{\pi \circ k_t\}_{0 \leq t \leq 1}$ is a homotopy between f and g. $\square$

Problem 4.

Let $X := \mathsf{S}^1 \times \mathsf{D}^2$ be the solid torus and $A := \mathsf{S}^1 \times \partial\mathsf{D}^2$ its boundary; then $X \cong \mathsf{S}^1$ and $A \cong \mathsf{T}^2$, so

$$\mathsf{H}_j(X) \cong \begin{cases} \mathbb{Z} & j = 0, 1, \\ 0 & \text{else}, \end{cases} \qquad \mathsf{H}_j(A) \cong \begin{cases} \mathbb{Z} & j = 0, \\ \mathbb{Z}^{\oplus 2} & j = 1, \\ \mathbb{Z} & j = 2, \\ 0 & \text{else}. \end{cases}$$

By the long exact sequence $\cdots \to \mathsf{H}_j(A) \to \mathsf{H}_j(X) \to \mathsf{H}_j(X, A) \to \mathsf{H}_{j-1}(A) \to \cdots$ for relative homology, we have

$$0 \to \mathsf{H}_3(X, A) \to \mathbb{Z} \to 0 \to \mathsf{H}_2(X, A) \xrightarrow{\delta_2} \mathbb{Z}^{\oplus 2} \xrightarrow{\iota_1} \mathbb{Z} \xrightarrow{\kappa_1} \mathsf{H}_1(X, A) \xrightarrow{\delta_1} \mathbb{Z} \xrightarrow{\iota_0} \mathbb{Z} \xrightarrow{\kappa_0} \mathsf{H}_0(X, A) \to 0$$

and we calculate the relative homologies as follows.

- Immediately, $\mathsf{H}_3(X, A) \cong \mathbb{Z}$.
- $\mathsf{H}_1(A)$ is generated by a lateral loop $[x] \in \mathsf{H}_1(\mathsf{S}^1)$ and a meridianal loop $[y] \in \mathsf{H}_1(\partial\mathsf{D}^2)$. The inclusion $\iota : A \hookrightarrow X$ maps x to the same lateral loop, so that $\iota_1([x])$ is the single generator of $\mathsf{H}_1(X) \cong \mathbb{Z}$, but includes y into the contractible component D^2, whereby $\iota_1([x]) = 1$ and $\iota_1([y]) = 0$. Thus we have $\mathsf{im}(\delta_2) \cong \mathsf{ker}(\iota_1) \cong \mathbb{Z}$, and also $\mathsf{ker}(\delta_2) \cong 0$, so $\mathsf{H}_2(X, A) \cong \mathbb{Z}$.
- By the above, $\mathsf{ker}(\kappa_1) \cong \mathsf{im}(\iota_1) \cong \mathbb{Z}$, and so $\mathsf{ker}(\delta_1) \cong \mathsf{im}(\kappa_1) \cong 0$. Moreover ι_0 is injective since it's induced by the inclusion $\iota : A \hookrightarrow X$ of path connected spaces, so $\mathsf{im}(\delta_1) \cong \mathsf{ker}(\iota_0) \cong 0$. Thus $\mathsf{H}_1(X, A) \cong 0$.
- We now have $\mathsf{ker}(\kappa_0) \cong \mathsf{im}(\iota_0) \cong \mathbb{Z}$ since $\mathsf{ker}(\iota_0) \cong 0$. Then $\mathsf{im}(\kappa_0) \cong 0$, and since κ_0 is surjective, then $\mathsf{H}_0(X, A) \cong 0$.

Hence $\mathsf{H}_j(X, A) \cong \begin{cases} 0 & j = 0, 1, \\ \mathbb{Z} & j = 2, 3, \\ 0 & \text{else}. \end{cases}$ □

Problem 5.

For each $1 \le j \le n$, let θ_j be an angular coordinate for the j-th S^1 component of $\mathsf{T}^n \cong \prod^n \mathsf{S}^1$. Then $\mathsf{d}\theta_j$ is a closed 1-form on T^n, and $f^*\mathsf{d}\theta_j$ is a closed 1-form on M, with $[f^*\mathsf{d}\theta_j] = 0 \in \mathsf{H}^1_{\mathsf{dR}}(M) \cong 0$. So $[(f^*\mathsf{d}\theta_1) \wedge \ldots \wedge (f^*\mathsf{d}\theta_n)] = 0 \in \mathsf{H}^n_{\mathsf{dR}}(M)$, and

$$0 = \int_M (f^*\mathsf{d}\theta_1) \wedge \ldots \wedge (f^*\mathsf{d}\theta_n) = \int_M f^*(\mathsf{d}\theta_1 \wedge \ldots \wedge \mathsf{d}\theta_n) = \mathsf{deg}(f) \underbrace{\int_{\mathsf{T}^n} \mathsf{d}\theta_1 \wedge \ldots \wedge \mathsf{d}\theta_n}_{\neq 0},$$

where the integral on the right is nonzero since $\mathsf{d}\theta_1 \wedge \ldots \wedge \mathsf{d}\theta_n$ is a volume form on T^n. □

Problem 6.

Remark. We can actually do this more generally. Let $m, n \in \mathbb{N}$, denote by $\mathsf{Mat}_{m\times n}(\mathbb{R})$ the vector space of all $m \times n$ matrices, and denote by $X \subset \mathsf{Mat}_{m\times n}(\mathbb{R})$ the subset of those matrices having rank $k \in \mathbb{N}$.

Denote by $X' \subset \mathsf{Mat}_{m\times n}(\mathbb{R})$ the submanifold of those block matrices $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ whose upper-left $k \times k$ block a is invertible. Any matrix $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in X'$, written in the form above, has rank k if and only if the product

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{m\times n} \underbrace{\begin{pmatrix} 1_{k\times k} & -a^{-1}b \\ 0 & 1_{(m-k)\times(m-k)} \end{pmatrix}}_{m\times m} = \underbrace{\begin{pmatrix} a & 0 \\ c & -ca^{-1}b + d \end{pmatrix}}_{n\times m}$$

has rank k, since the matrix we're multiplying by is invertible. Since a already has rank k, this requires the lower-right $(n-k)\times(m-k)$ block $-ca^{-1}b+d$ of the matrix on the right-hand side to be 0. Thus the space X'' of rank-k matrices belonging to X' can be identified with $f^{-1}(0)$, where f is the smooth map

$$f : X' \to \mathsf{Mat}_{(n-k)\times(m-k)}(\mathbb{R}), \quad f(x) := -ca^{-1}b + d,$$

with a, b, c, d corresponding to x as above. To conclude the proof, it's enough to show that X'' is a manifold, since matrices in X and matrices in X'' are related by (smooth) elementary row operations. Now, it's enough to check that 0 is a regular value of f. For any $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in f^{-1}(0)$, if $y \in \mathsf{Mat}_{(n-k)\times(m-k)}(\mathbb{R})$ is arbitrary, then defining

$$\alpha : [0,1] \to \mathsf{Mat}_{(n-k)\times(m-k)}(\mathbb{R}), \quad \alpha(t) := \begin{pmatrix} a & b \\ c & d \end{pmatrix} + t\begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix},$$

we see that

$$\mathsf{d}f_x \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix} = (f \circ \alpha)'(0) = (-ca^{-1}b + d + ty)'(0) = y,$$

whereby $\mathsf{d}f_x$ is surjective. Thus X'' is a submanifold of X', and in particular is a manifold. □

Problem 7.

Suppose $\omega \in \Omega^1(\mathsf{S}^2)$ has $\phi^*\omega = \omega$ for every $\phi \in \mathsf{SO}(3)$. Then for arbitrary $x \in \mathsf{S}^2$ and $v \in \mathsf{T}_x\mathsf{S}^2$,

$$\omega_x(v) = \phi^*\omega_x(v) = \omega_{\phi(x)} \circ \mathsf{d}\phi_x(v)$$

for every $\phi \in \mathsf{SO}(3)$ by the definition of the pullback. Now $\mathsf{SO}(3)$ acts transitively on TS^2 by $\phi \cdot (y, w) := (\phi(y), \mathsf{d}\phi_y(w))$, so we may let $\phi \in \mathsf{SO}(3)$ above be such that $\phi \cdot (x, v) = (x, 0)$, and thus $\omega_x(v) = 0$. Hence $\omega \equiv 0$. □