1. $\mathbb{R}^n - \{m \text{ points}\} \overset{h.e.}{\simeq} \bigvee_{i=1}^{m} S^{n-1}$

2. $\pi_1(\mathbb{RP}^2 \times \mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \Rightarrow \exists$ 4 covers, ~~2 subgroups are con~~

3. Stokes, volume forms have nonzero integral.

4. $M \overset{h.e.}{\simeq}$ surface of genus $4 \setminus pt \Rightarrow \chi(M) = 2 - 2\cdot 4 - 1 = -7 \Rightarrow \int_M K \, dA = -14\pi$.

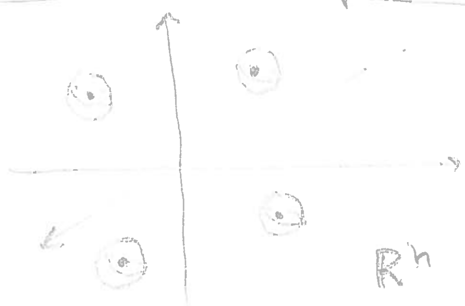
5. Mayer Vietoris / short Exact Sequence

6. Identify dual of T_pM w/ itself, use tangent bundle trick, ie, tangent maps of coordinate changes are block matrices.

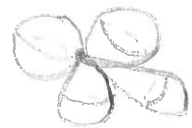
7. $\ker d_0 = 0$ because the only constant functions w/ compact support is 0. $\operatorname{Im} d_0 = \ker d_1$.

GeoTop - Fall 05

(1.) *The complement of a finite set of pts in $\mathbb{R}^n$ is simply-connected if $n \geq 3$*



$\simeq$



$\simeq \vee^k S^{n-1}$

See that $\mathbb{R}^n \setminus \{p_1, \dots, p_k\} \simeq \vee^k S^{n-1}$, a bouquet of k $(n-1)$-spheres.
Then $\pi_1(\mathbb{R}^n \setminus \{p_1, \dots, p_k\}) \cong \pi_1(\vee^k S^{n-1}) \cong *^k \pi_1(S^{n-1})$

by Seifert - van Kampen.

But for $n \geq 3 \Rightarrow (n-1) \geq 2$, hence $\pi_1(S^{n-1}) = 1$,
hence $\pi_1(\mathbb{R}^n \setminus \{p_1, \dots, p_k\}) \cong *^k 1 \cong 1 \Rightarrow$ simply connected.

(2.) Recall that $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$: now describe the equivalence classes of connected covers of $\mathbb{R}P^2 \times \mathbb{R}P^2$.

Recall equiv. classes of connected covers of $\mathbb{R}P^2 \times \mathbb{R}P^2$ correspond to subgroups of $\pi_1(\mathbb{R}P^2 \times \mathbb{R}P^2) \cong \pi_1(\mathbb{R}P^2) \times \pi_1(\mathbb{R}P^2)$

$$\cong \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

This group has subgroups 1, $\mathbb{Z}_2 \oplus 1$, $1 \oplus \mathbb{Z}_2$, and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$

- $1 \oplus 1$ corresponds to universal cover $S^2 \times S^2$
- $1 \oplus \mathbb{Z}_2$, $\mathbb{Z}_2 \oplus 1$ correspond to the universal covers of the coordinate spaces: $S^2 \times \mathbb{R}P^2$ & $\mathbb{R}P^2 \times S^2$ respectively.
- $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ corresponds to the trivial cover $\mathbb{R}P^2 \times \mathbb{R}P^2$.

3.) $\alpha \in \Omega^2(S^4)$ closed. Show that $\alpha \wedge \alpha = 0$ at some $p \in S^4$.

See that $\alpha \wedge \alpha \in \Omega^4(S^4)$ and
$$d(\alpha\wedge\alpha) = d\alpha \wedge \alpha + \alpha \wedge d\alpha = 0 \wedge \alpha + \alpha \wedge 0 = 0$$

hence $\alpha\wedge\alpha$ is closed.

Suppose that $\alpha \wedge \alpha \neq 0$ everywhere on S^4, hence a volume form. Now apply Stokes:
$$\int_{S^4} \alpha\wedge\alpha = \int_{B^5} d(\alpha\wedge\alpha) = \int_{B^5} 0 = 0$$
which is a contradiction since volume forms have non-zero integral, hence $\alpha\wedge\alpha = 0$ somewhere on S^4.

4.) $M =$ 

$\subseteq \mathbb{R}^3$

Compute $\int_M K\,dA$ where K = Gaussian curvature

See that $M \simeq$  $\simeq$ 

$\simeq$  $\simeq M_4 \setminus \{pt\}$.

removing a point adds a 1-cell!

Then $\chi(M) = \chi(M_4 \setminus pt) = (2-2g)-1 = (2-8)-1 = -7$

and then by Gauß-Bonnet:
$$\int_M K\,dA = 2\pi\chi(M) = 2\pi(-7) = -14\pi$$

(5.) For any X, $H_i(X\times S^1) \cong H_i(X)\oplus H_{i-1}(X)$

$X\times S^1 = X\times \bigcirc$, so Let $A = X\times C$, $B = X\times \supset$

$A\cap B = X\times \overset{\frown}{\smile} = X \amalg X$

Now apply Mayer-Vietoris:

$$\cdots\to H_{i+1}(A)\oplus H_{i+1}(B) \longrightarrow H_{i+1}(X\times S^1) \to H_i(A\cap B) \to H_i(A)\oplus H_i(B)$$
$$\longrightarrow H_i(X\times S^1) \to H_{i-1}(A\cap B) \longrightarrow \cdots$$

Now consider the map (for any i)

$$H_i(A\cap B) \cong H_i(X\times \overset{\frown}{\smile}) \xrightarrow{(i^*,j^*)} H_i(A)\oplus H_i(B)$$
$$\cong H_i(X\times C)\oplus H_i(X\times \supset)$$

induced from $i: A\cap B \hookrightarrow A$, $j: A\cap B \hookrightarrow B$.

So:

$i^*: H_i(A\cap B) \longrightarrow H_i(A)$ $\qquad$ $j^*: H_i(A\cap B) \longrightarrow H_i(B)$

$X\times \overset{\frown}{\smile} \xrightarrow[i]{} X\times C$ $\qquad$ $X\times \overset{\frown}{\smile} \xhookrightarrow[j]{} X\times \supset$

Clearly an i-chain in $X\times \overset{\frown}{\smile}$ is the same in $X\times C$ & $X\times \supset$ (and cannot, say, become a boundary). Hence both are identity, hence

$$(i^*,j^*): \underbrace{H_i(A\cap B)}_{\cong H_i(X)\oplus H_i(X)} \longrightarrow \underbrace{H_i(A)\oplus H_i(B)}_{\cong H_i(X)\oplus H_i(X)} \quad \text{for all } i,$$
$$\alpha \longmapsto (\alpha,\alpha)$$

i.e. $\underline{\operatorname{im}(i^*,j^*)_i = \langle(\alpha,\alpha)\rangle = H_i(X)}$

Therefore, we now have the sequence:

$$\longrightarrow H_{i+1}(A\cap B) \xrightarrow{(i^*,j^*)_i} H_i(A)\oplus H_i(B) \longrightarrow H_i(X\times S^1) \xrightarrow{\partial_i} H_{i-1}(A\cap B)$$
$$\xrightarrow{(i^*,j^*)_{i-1}} H_{i-1}(A)\oplus H_{i-1}(B) \longrightarrow H_{i-1}(X\times S^1) \longrightarrow \cdots$$

$$\Longrightarrow \cdots \longrightarrow H_i(X)\oplus H_i(X) \xrightarrow{(i^*,j^*)_i} H_i(X)\oplus H_i(X) \xrightarrow{\phi} H_i(X\times S^1) \xrightarrow{\partial_i} H_{i-1}(X)\oplus H_{i-1}(X)$$
$$\xrightarrow{(i^*,j^*)_{i-1}} H_{i-1}(X)\oplus H_{i-1}(X) \longrightarrow \cdots$$

Now, $im(i^*,j^*)_i = H_i(X)$ for all i, hence:

$$H_i(X) = im(i^*,j^*)_i = \ker\phi \Rightarrow im\,\phi = H_i(X)$$
$$\Rightarrow \underline{\ker \partial_i = H_i(X)}$$

and $im(i^*,j^*)_{i-1} = H_{i-1}(X) \Rightarrow \ker(i^*,j^*)_{i-1} = H_{i-1}(X)$

$\Rightarrow \underline{im\,\partial_i = H_{i-1}(X)}$

Hence $\underline{H_i(X\times S^1) \cong H_{i-1}(X)\oplus H_i(X)}$

(7.) Find $H^1_c(\mathbb{R})$, the de Rham cohomology w/ compact support, from the definition !

We have the chain complex

$$0 \longrightarrow \Omega^0_c(\mathbb{R}) \xrightarrow{d_0} \Omega^1_c(\mathbb{R}) \xrightarrow{d_1} 0$$

and we know $H^0_c(\mathbb{R}) = \ker d_0$, $H^1_c(\mathbb{R}) = \ker d_1 / \operatorname{im} d_0$.

$H^0_c(\mathbb{R})$: Recall that $\Omega^0_c(\mathbb{R}) = C^\infty_c(\mathbb{R}) \subseteq C^\infty(\mathbb{R}) = \Omega^0(\mathbb{R})$,
and recall that $\ker(d_0 : \Omega^0(\mathbb{R}) \to \Omega^1(\mathbb{R})) = \{f \in \Omega^0(\mathbb{R}) : d_0 f = 0\}$
$= \{\text{constant functions}\}$

But the only constant function with compact support is 0,
hence $\ker(d_0 : \Omega^0_c(\mathbb{R}) \to \Omega^1_c(\mathbb{R})) \cong \{0\}$, hence $\underline{H^0_c(\mathbb{R}) \cong 0}$.

$H^1_c(\mathbb{R})$: For $f \in \Omega^0_c(\mathbb{R})$, i.e. smooth fn. with compact support,
we know $\exists M > 0$ s.t. $|x| > M \Rightarrow f(x) = 0$. Therefore,

$$\int_{\mathbb{R}} df = \int_{\mathbb{R}} f' dx = \lim_{x \to \infty} f(x) - \lim_{x \to -\infty} f(x) = 0, \text{ hence}$$

$$\operatorname{im} d_0 \subseteq \{ w = g(x)dx : g \text{ cpt support } \& \int_{\mathbb{R}} g = 0 \} \subseteq \Omega^1_c(\mathbb{R}).$$

Choose g s.t. g has cpt supp. & $\int_{\mathbb{R}} g = 0$. Now consider the function $h(x) = \int_{-\infty}^x g(t)dt$; clearly $dh = g(x)dx$, hence if h has compact support then $\operatorname{im} d_0 = \{ g(x)dx : g \text{ cpt supp}, \int_{\mathbb{R}} g = 0\}$

Since g has compact support, $\exists M$ s.t. $|x| > M \Rightarrow g(x) = 0$.
So then $\int_{-\infty}^{-M} g(t)dt = 0$ & $\int_M^\infty g(t)dt = 0$

$\Rightarrow h(x) = \underbrace{\int_{-\infty}^{-M} g(t)dt}_{\to 0} + \int_{-M}^x g(t)dt$; hence h nonzero only in $[-M, M]$,
i.e. h has compact support

Now consider the linear map $I : \Omega^1_c(\mathbb{R}) \to \mathbb{R}$
$$\omega \longmapsto \int_{\mathbb{R}} \omega$$

Clearly $\ker I = \{\omega = g\,dx : g \text{ cpt support } \& \int_{\mathbb{R}} g = 0\}$
$$= \operatorname{im} d_0$$

Hence by isomorphism theorem:

$$\Omega^1_c(\mathbb{R}) / \operatorname{im} d_0 \cong \mathbb{R}, \text{ i.e.:}$$

$$H^1_c(\mathbb{R}) = \ker d_1 / \operatorname{im} d_0 = \Omega^1_c(\mathbb{R}) / \operatorname{im} d_0 \cong \mathbb{R}.$$

$$\Rightarrow \underline{\underline{H^1_c(\mathbb{R}) \cong \mathbb{R}}}$$