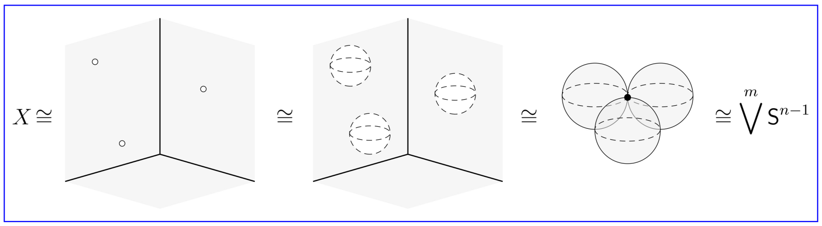
2005, Fall

Problem 1.

Let $n \geq 3$ and set $X := \mathbb{R}^n \setminus \{x_1, \dots, x_m\}$, with $x_i \neq x_j$ if $i \neq j$. Expanding each missing point into a small bubble, we may shrink the complement of these bubbles to a point to obtain a wedge of m copies of S^{n-1}.



Thus $\pi_1(X) \cong \pi_1\left(\bigvee^m \mathsf{S}^{n-1}\right) \cong \pi_1(\mathsf{S}^{n-1})^{*m} \cong 1$. □

Problem 2.

Equivalence classes of connected covers of $\mathbb{R}\mathsf{P}^2 \times \mathbb{R}\mathsf{P}^2$ are in bijection with the 4 subgroups of

$$\pi_1(\mathbb{R}\mathsf{P}^2 \times \mathbb{R}\mathsf{P}^2) \cong \pi_1(\mathbb{R}\mathsf{P}^2) \times \pi_1(\mathbb{R}\mathsf{P}^2) \cong \mathbb{Z}_2^{\oplus 2}.$$

The identity subgroup corresponds to the universal cover $\mathsf{S}^2 \times \mathsf{S}^2$; the entire group corresponds to the trivial cover $\mathbb{R}\mathsf{P}^2 \times \mathbb{R}\mathsf{P}^2$; the subgroups generated by $(0,1)$ and $(1,0)$ correspond to the covers $\mathsf{S}^2 \times \mathbb{R}\mathsf{P}^2$ and $\mathbb{R}\mathsf{P}^2 \times \mathsf{S}^2$, respectively. □

Problem 3.

Assume $(\alpha \wedge \alpha)_x \neq 0$ for all $x \in \mathsf{S}^4$. Then $\alpha \wedge \alpha$ is a volume form on S^4, so $\int_{\mathsf{S}^4} \alpha \wedge \alpha \neq 0$. But

$$\int_{\mathsf{S}^4} \alpha \wedge \alpha = \int_{\mathsf{B}^5} \mathsf{d}(\alpha \wedge \alpha) = \int_{\mathsf{B}^5} [\underbrace{(\mathsf{d}\alpha)}_{=0} \wedge \alpha + \alpha \wedge \underbrace{(\mathsf{d}\alpha)}_{=0}] = 0,$$

by Stokes, a contradiction. □

Problem 4 (?)**.**

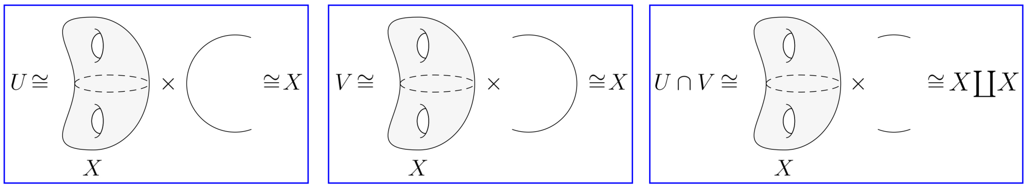
The Euler characteristic of M is that of a genus-3 surface, $\chi(M) = -4$. And, ∂M is the boundary of the plane, so integrating the geodesic curvature k_{g} over this boundary yields the sum of its exterior angles, namely $4(3\pi/2) = 6\pi$. Then

$$\iint_M K \mathsf{d}A = 2\pi\chi(M) - \int_{\partial M} k_{\mathsf{g}} \mathsf{d}s = 2\pi(-4) - 6\pi = -14\pi$$

by Gauss-Bonnet. □

Problem 5 (?)**.**

Defining



gives $U \cup V \cong X \times \mathsf{S}^1$. For any $j \in \mathbb{Z}$, by Mayer-Vietoris we have an exact sequence

$$\mathsf{H}_j(X)^{\oplus 2} \xrightarrow{\iota_j} \mathsf{H}_j(X)^{\oplus 2} \xrightarrow{f} \mathsf{H}_j(X \times \mathsf{S}^1) \xrightarrow{\partial} \mathsf{H}_{j-1}(X)^{\oplus 2} \xrightarrow{\iota_{j-1}} \mathsf{H}_{j-1}(X)^{\oplus 2}.$$

Here, $\iota_j = ((\iota_U)_*, (\iota_V)_*)$ is the map induced by the inclusions $\iota_U : U \cap V \hookrightarrow U$ and $\iota_V : U \cap V \hookrightarrow V$, and acts as $\iota_j([\omega]) = ([\omega], [\omega])$ on any $[\omega] \in \mathsf{H}_j(X \coprod X) \cong \mathsf{H}_j(X)^{\oplus 2}$. So $\mathsf{im}(\iota_j) \cong \mathsf{H}_j(X)$ and consequently $\mathsf{ker}(\iota_j) \cong \mathsf{H}_j(X)$ also. We have a similar result for $j-1$. Then

$$\mathsf{im}(\partial) \cong \mathsf{ker}(\iota_{j-1}) \cong \mathsf{H}_{j-1}(X), \quad \mathsf{ker}(\partial) \cong \mathsf{im}(f) \cong \mathsf{H}_j(X)$$

since $\mathsf{ker}(f) \cong \mathsf{im}(\iota_j) \cong \mathsf{H}_j(X)$, and therefore $\mathsf{H}_j(X \times \mathsf{S}^1) \cong \mathsf{H}_j(X) \oplus \mathsf{H}_{j-1}(X)$. □

Problem 6.

- Let M be an m-manifold. Via the canonical projection $\pi : \mathsf{T}^*M \twoheadrightarrow M$, we may lift any chart (U, φ) of M to a subset $\mathsf{T}^*U := \pi^{-1}(U) \subset \mathsf{T}^*M$ and a map $\varphi \times \mathsf{d}\varphi : \mathsf{T}^*U \to \varphi(U) \times \mathbb{R}^m$. Requiring this map to be a homeomorphism defines a topology on T^*M and makes $(\mathsf{T}^*U, \varphi \times \mathsf{d}\varphi)$ into a chart for T^*M. This topology inherits the second countable and Hausdorff properties from that of M. Moreover, given any two charts $(\mathsf{T}^*U_1, \varphi_1 \times \mathsf{d}\varphi_1), (\mathsf{T}^*U_2, \varphi_2 \times \mathsf{d}\varphi_2)$ with $\mathsf{T}^*U_1 \cap \mathsf{T}^*U_2 \neq \varnothing$, we have $U_1 \cap U_2 \neq \varnothing$ and so the transition map

  $$\tau := (\varphi_2 \times \mathsf{d}\varphi_2) \circ (\varphi_1 \times \mathsf{d}\varphi_1)^{-1} = (\varphi_2 \circ \varphi_1^{-1}) \times (\mathsf{d}\varphi_2 \circ \mathsf{d}\varphi_1^{-1})$$

  is smooth since its first component is a transition map of M and its second component is linear. Therefore T^*M is a (smooth) manifold.

- It remains to check that any transition map τ as above is orientation-preserving. Say T^*U_1 has local coordinates $(x_1, \ldots, x_m, v_1, \ldots, v_m)$, and express the differential $\mathsf{d}\tau_{(x,v)}$ at a point $(x, v) \in \mathsf{T}^*U_1$ as a $(2m) \times (2m)$ block matrix.

  (i) In the upper-left block, we differentiate the first m entries of τ w.r.t. x and obtain the usual Jacobian $\mathsf{d}(\varphi_2 \circ \varphi_1^{-1})_x$.

  (ii) In the upper-right block, we differentiate the first m entries of τ, which are independent of v, w.r.t. v, and obtain a 0 block.

  (iii) In the lower-right block, we differentiate the last m entries of τ w.r.t. v. For $1 \le i, j \le m$, the ij-entry in this block is

  $$\partial_{v_j}(\mathsf{d}\varphi_2 \circ \mathsf{d}\varphi_1^{-1})_i = \partial_{v_j} \sum_{k=1}^{m} \partial_{x_k}(\varphi_2 \circ \varphi_1^{-1})_i(x) v_k = \partial_{x_j}(\varphi_2 \circ \varphi_1^{-1})_i(x),$$

  which coincides with the ij-entry of the Jacobian $\mathsf{d}(\varphi_2 \circ \varphi_1^{-1})_x$.

Hence

$$\mathsf{det}(\mathsf{d}\tau_{(x,v)}) = \begin{vmatrix} \mathsf{d}(\varphi_2 \circ \varphi_1^{-1})_x & 0 \\ * & \mathsf{d}(\varphi_2 \circ \varphi_1^{-1})_x \end{vmatrix} = \mathsf{det}\left(\mathsf{d}(\varphi_2 \circ \varphi_1^{-1})_x\right)^2 > 0$$

as desired.

□

Problem 7.

Background. This problem concerns *compactly supported de Rham cohomology.* Given a manifold M^m and $0 \leq k \leq m$, we use this cohomology to define the *cup product,*

$$\smile : \mathsf{H}^k(M) \times \mathsf{H}_{\mathsf{c}}^{m-k}(M) \to \mathbb{R},$$

given by $[\alpha] \smile [\beta] := \int_M \alpha \wedge \beta$. Poincaré duality then states that the map $\mathsf{H}^k(M) \to \mathsf{H}_{\mathsf{c}}^{m-k}(M)^*$ given by $[\alpha] \mapsto [\alpha] \smile (\cdot)$ is an isomorphism.

Denote by $\mathsf{H}_{\mathsf{c}}^\bullet(\mathbb{R})$ the cohomology of $(\Omega_{\mathsf{c}}^\bullet(\mathbb{R}), \mathsf{d}^\bullet)$. For all $j \geq 2$ we clearly have $\mathsf{H}_{\mathsf{c}}^j(\mathbb{R}) \cong 0$ since $\Omega_{\mathsf{c}}^j(\mathbb{R}) \cong 0$, so it remains to compute $\mathsf{H}_{\mathsf{c}}^j(\mathbb{R})$ for $j = 0, 1$. Observe that both $\Omega_{\mathsf{c}}^0(\mathbb{R})$ and $\Omega_{\mathsf{c}}^1(\mathbb{R})$ are canonically isomorphic to $\mathsf{C}_{\mathsf{c}}^\infty(\mathbb{R})$, the subset of $\mathsf{C}^\infty(\mathbb{R})$ consisting of compactly supported functions.

- Note that $f \in \mathsf{H}_{\mathsf{c}}^0(\mathbb{R}) \cong \mathsf{ker}(\mathsf{d}^0)$ if and only if f is constant, whereby $f \equiv 0$ since no other such function is compactly supported. Thus $\mathsf{H}_{\mathsf{c}}^0(\mathbb{R}) \cong 0$.
- Consider the map $I : \mathsf{H}_{\mathsf{c}}^1(\mathbb{R}) \cong \mathsf{C}_{\mathsf{c}}^\infty(\mathbb{R}) \to \mathbb{R}$ given by $I(f) := \int_{\mathbb{R}} f$. Note that $\mathsf{ker}(I) \subset \mathsf{im}(\mathsf{d}^0)$ since if $f \in \mathsf{ker}(I)$, then $f \cong \mathsf{d}g$ where g is the compactly supported function given by $g(x) := \int_{-\infty}^x f(x)\mathsf{d}x$. Conversely if $f \in \mathsf{im}(\mathsf{d}^0)$, then $f = \mathsf{d}g$ for some $g \in \mathsf{C}_{\mathsf{c}}^\infty(\mathbb{R})$, and

$$\int_{\mathbb{R}} f = \int_{-\infty}^{\infty} g'(x)\mathsf{d}x = \lim_{x\to\infty} g(x) - \lim_{x\to-\infty} g(x) = 0,$$

so this shows $\mathsf{im}(\mathsf{d}^0) \cong \mathsf{ker}(I)$. Then $\mathsf{H}_{\mathsf{c}}^1(\mathbb{R}) \cong \mathsf{ker}(\mathsf{d}^1)/\mathsf{im}(\mathsf{d}^0) \cong \mathsf{C}_{\mathsf{c}}^\infty(\mathbb{R})/\mathsf{ker}(I) \cong \mathbb{R}$, since I is clearly surjective.

In summary, $\mathsf{H}_{\mathsf{c}}^j(\mathbb{R}) \cong \begin{cases} \mathbb{R} & j = 1, \\ 0 & \text{else.} \end{cases}$ □