Qualifying Exam in Geometry/Topology Fall 2005

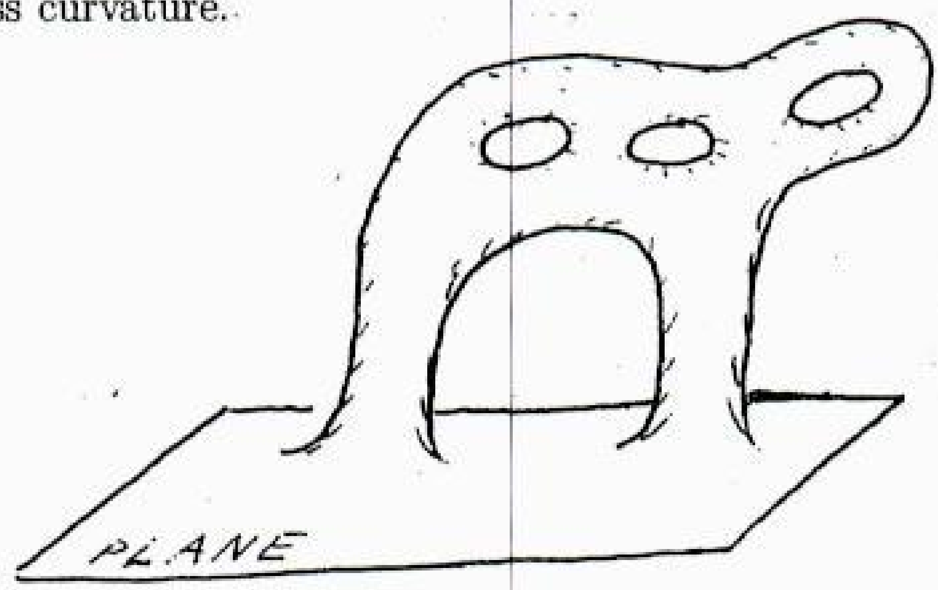
Solve all **SEVEN** *problems. Partial credit will be given to partial solutions.*

1. Show that the complement of a finite set of points in R^n is simply connected if $n \geq 3$.

2. Fix a space X and say that two covers $p_i : \tilde{X}_i \to X$, for $i = 1, 2$, are *equivalent* if there is a homeomorphism $f : \tilde{X}_1 \to \tilde{X}_2$ so that $p_1 = p_2 \circ f$. Recall that real projective 2-space RP^2 has its fundamental group isomorphic to the integers mod two, and describe the equivalence classes of connected covers of $RP^2 \times RP^2$.

3. Let α be a closed 2-form on $S^4 = \{(x_1, \ldots, x_5) \in R^5 : x_1^2 + \cdots + x_5^2 = 1\}$. Show that $\alpha \wedge \alpha = 0$ at some point $p \in S^4$.

4. Consider the surface $M \subset R^3$ pictured below. Compute the integral $\int_M K dA$, where K is the Gauss curvature.



(picture of the surface M)

5. Show that for any space X, we have $H_i(X \times S^1) \approx H_i(X) \oplus H_{i-1}(X)$, where S^1 denotes the circle.

6. Given a smooth manifold M, define the *cotangent bundle* $T^*(M)$ to be the set of all pairs (p, q), where $p \in M$ and q lies in the dual vector space to the tangent space $T_p(M)$ of M at p. Show that $T^*(M)$ has the structure of a smooth orientable manifold. (Do not assume that M itself is orientable.)

7. Let M be a smooth manifold. Let $\Omega_c^i(M) \subset \Omega^i(M)$ be the set of smooth i-forms with compact support, i.e., $\omega \in \Omega_c^i(M)$ is zero outside a compact set. Then there is a chain complex

$$0 \to \Omega_c^0(M) \xrightarrow{d_0} \Omega_c^1(M) \xrightarrow{d_1} \Omega_c^2(M) \xrightarrow{d_2} \cdots,$$

where d is the exterior derivative restricted to forms with compact support. Define the ith de Rham cohomology of M *with compact support* to be $ker(d_i)/im(d_{i-1})$. Compute the ith de Rham cohomology of the real line R with compact support for all $i \geq 0$. (Your answer will differ from the usual de Rham cohomology of R.)