7. Let f be a homogeneous polynomial in k (real) variables. Homogeneity means that there is some positive integer m for which

$$f(tx_1, \ldots, tx_k) = t^m f(x_1, \ldots, x_k),$$

for all $t \in \mathbf{R}$ and $x_1, \ldots, x_k \in \mathbf{R}$. Prove that the set of points $x \in \mathbf{R}^k$ for which $f(x) = a$ is a $(k-1)$-dimensional submanifold of $\mathbf{R}^k$, provided $a \neq 0$. [Hint: Use Euler's identity for homogeneous polynomials, which states that $\sum_{i=1}^k x_i \frac{\partial f}{\partial x_i} = m \cdot f$.]

1. Stokes, $[\omega] \mapsto \int_M \omega$ is an isomorphism $H^n_{dR}(M^n) \to \mathbb{R}$ if M^n is connected.

2. $\int_E \omega = \int_{S^2} \omega$ because $d\omega = 0$ in between E and S^2.

3. Van Kampen $\langle a, b \mid a b^2 a^{-1} b^{-2} \rangle$

4. $\int_M k \, dA = 2\pi \chi(M)$

5. $X \overset{h.e}{\simeq} S^1 \times S^1$

6. $f: S^2 \to S^3$, im f has no regular values $\Rightarrow$ im f has measure 0.

7. Suppose $f(x) = a$. Then $T_x f(x) = a \cdot m \neq 0 \Rightarrow$ Regular Value Thm.

(1.) $\omega \in \Omega^k(T^k)$ exact $\iff \int_{T^k}\omega = 0$

($\Rightarrow$) Suppose ω exact. Then $\exists\, \alpha \in \Omega^{k-1}(T^k)$ s.t. $d\alpha = \omega$.
Now apply Stokes:

$$\int_{T^k}\omega = \int_{\partial(\text{Solid } T^k)}\omega = \int_{\text{Solid } T^k} d\omega = \int_{\text{Solid } T^k} d(d\alpha) = \int_{\text{Solid } T^k} 0 = 0$$

($\Leftarrow$) Suppose $\int_{T^k}\omega = 0$. Since $\dim T^k = k$, we have $\Omega^{k+1}(T^k) = 0$, hence $d\omega = 0$, hence $[\omega] \in H^k(T^k)$ well-defined.

Now, since T^k open, connected, orient., consider the top-dim'l de Rham iso. $I: H^k(T^k) \to \mathbb{R}$, $\omega \mapsto \int_{T^k}\omega$

Since $\int_{T^k}\omega = 0$, $[\omega] = 0 \Rightarrow$ <u>ω exact</u>

(2.) $\omega \in \Omega^{n-1}(\mathbb{R}^n)$ given by
$$\omega = \frac{\sum_{i=1}^n (-1)^{i+1} dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n}{\left(\sum_{i=1}^n x_i^2\right)^{n/2}}$$

(a) Show ω closed in $\mathbb{R}^n \setminus \{0\}$

(b) Compute $\int_E \omega$ where $E = \{ \frac{x_1^2}{9} + \sum_{i=2}^n x_i^2 = 2007 \}$

(a)
$$d\omega = \sum_{i=1}^n (-1)^{i+1} d\left(\frac{x_i}{\left(\sum_{j=1}^n x_j^2\right)^{n/2}}\right) \wedge dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n$$

and:
$$d\left(\frac{x_i}{(\sum x_j^2)^{n/2}}\right) = \sum_{k=1}^n \frac{\partial}{\partial x_k}\left[x_i \left(\sum_{j=1}^n x_j^2\right)^{-n/2}\right] dx_k$$

see that:
$$\frac{\partial}{\partial x_k}\left(x_i (\textstyle\sum x_j^2)^{-n/2}\right) = 2x_k x_i \frac{-n}{2}\left(\textstyle\sum x_j^2\right)^{\frac{-n-2}{2}} = \frac{-x_k x_i n}{(\sum x_j^2)^{\frac{n+2}{2}}} \quad k \neq i$$

$$\frac{\partial}{\partial x_i}\left(x_i (\textstyle\sum x_j^2)^{-n/2}\right) = \left(\textstyle\sum x_j^2\right)^{-n/2} + x_i\left(\frac{-n}{2}\right) 2 x_i \left(\textstyle\sum x_j^2\right)^{\frac{-n-2}{2}}$$
$$= \underline{\frac{1}{(\sum x_j^2)^{n/2}} + \frac{-x_i^2 n}{(\sum x_j^2)^{\frac{n+2}{2}}}}$$

So:

contains $dx_k \wedge dx_k$ $\forall k \neq i$

$$d\left(\frac{x_i}{(\sum x_j^2)^{n/2}}\right) \wedge dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n = \underbrace{\cancel{\left(\sum_{k \neq i} \frac{-x_k x_i n}{(\sum x_j^2)^{\frac{n+2}{2}}} dx_k\right) \wedge dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n}}_{0}$$
$$+ \left(\frac{1}{(\sum x_j^2)^{n/2}} + \frac{-x_i^2 n}{(\sum x_j^2)^{\frac{n+2}{2}}}\right) dx_i \wedge dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n$$

hence $d\omega = \sum_{i=1}^{n} (-1)^{i+1} \left(\frac{1}{(\sum x_j^2)^{n/2}} + \frac{-x_i^2 n}{(\sum x_j^2)^{\frac{n+2}{2}}} \right) \underline{dx_i} \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$

$= \sum_{i=1}^{n} (-1)^{i+1} \left(\frac{1}{(\sum x_j^2)^{n/2}} + \frac{-x_i^2 n}{(\sum x_j^2)^{n+2/2}} \right) (-1)^{i-1} dx_1 \wedge \cdots \wedge \underline{dx_i} \wedge \cdots \wedge dx_n$

$= \sum_{i=1}^{n} (-1)^{2i} \left(\frac{1}{(\sum x_j^2)^{n/2}} + \frac{-x_i^2 n}{(\sum x_j^2)^{\frac{n}{2}+1}} \right) dx_1 \wedge \cdots \wedge dx_n$

$= \sum_{i=1}^{n} \left(\frac{(\sum x_j^2) + -x_i^2 n}{(\sum x_j^2)^{n/2+1}} \right) dx_1 \wedge \cdots \wedge dx_n$

$= \frac{1}{(\sum x_j^2)^{n/2+1}} \cdot \left(\sum_{i=1}^{n} (\sum x_j^2) - \sum_{i=1}^{n} x_i^2 n \right) dx_1 \wedge \cdots \wedge dx_n$

$= \frac{1}{(\sum x_j^2)^{\frac{n+2}{2}}} \cdot \left(\cancel{n \sum_{j=1}^{n} x_j^2 - n \sum_{i=1}^{n} x_i^2}^{\,0} \right) dx_1 \wedge \cdots \wedge dx_n = 0$

(b) $E = \{ \frac{x_1^2}{9} + \sum_{i=2}^{n} x_i^2 = 2004 \}$; find $\int_E \omega$.

ω is closed on all of $\mathbb{R}^n \setminus \{0\}$ and both $E, S^{n-1} \subseteq \mathbb{R}^n \setminus \{0\}$

hence ω is closed on both, hence $[\omega]$ well-defined in $H^{n-1}(E)$ & $H^{n-1}(S^{n-1})$; but $E \simeq S^{n-1}$ (homotopy equiv),

hence $H^{n-1}(E) \cong H^{n-1}(S^{n-1})$, hence $\int_E \omega = \int_{S^{n-1}} \omega$ by top-dim de Rham iso.

So $\int_E \omega = \int_{S^{n-1}} \omega = \int_{S^{n-1}} \frac{\sum_{i=1}^{n} (-1)^{i+1} x_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n}{(\sum_{i=1}^{n} x_i^2)^{n/2}}$

$= \int_{S^{n-1}} \sum_{i=1}^{n} (-1)^{i+1} x_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$ $\rightarrow$ apply Stokes

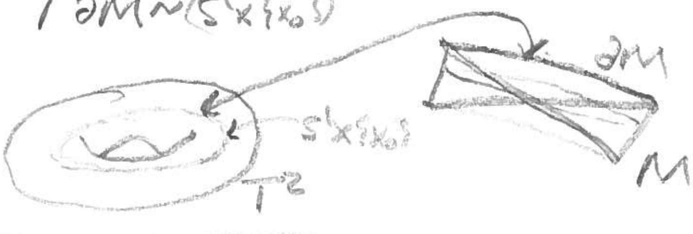
$= \int_{B^n} \sum_{i=1}^{n} (-1)^{i+1} dx_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$

$= \int_{B^n} \sum_{i=1}^{n} (-1)^{i+1} (-1)^{i-1} dx_1 \wedge \cdots \wedge dx_n = \int_{B^n} \sum_{i=1}^{n} dx_1 \wedge \cdots \wedge dx_n$

$= n \int_{B^n} dx_1 \wedge \cdots \wedge dx_n = \underline{n \operatorname{vol}(B^n)}$

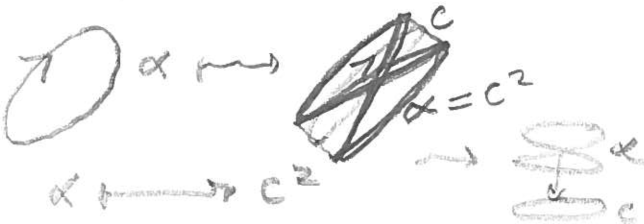
(3.) $X = S^1 \times S^1 \amalg M / \partial M \sim (S^1 \times \{x_0\})$

(a) $\pi_1(X)$
(b) $H_i(X)$



(a) Let $A = M$, $B = T^2$, $A \cap B = \partial M \cong S^1$, $A \cup B = X$
and consider the inclusions $i_1 : A\cap B \hookrightarrow A$, $i_2 : A \cap B \hookrightarrow B$

$$i_1^* : \pi_1(A\cap B) \longrightarrow \pi_1(A) \qquad i_2 : \pi_1(A\cap B) \longrightarrow \pi_1(B)$$
$$\mathbb{Z}\langle\alpha\rangle \cong \quad \mathbb{Z}\langle c\rangle \cong \qquad\qquad \mathbb{Z}\langle\alpha\rangle \cong \quad \mathbb{Z}\langle a\rangle \oplus \mathbb{Z}\langle b\rangle \cong$$



$\alpha \longmapsto c^2$ $\qquad\qquad$ $\alpha \longmapsto a$

→ clearly the boundary circle $A \cap B \sim \partial M$ wraps around the generating circle c twice

⟹ the circle $A \cap B \sim S^1 \times \{x_0\}$ clearly corresponds to the generating circle a.

Now apply Seifert-van Kampen:

$$\pi_1(X) = \pi_1(A) * \pi_1(B) / \langle i_1^*(\alpha)\, i_2^*(\alpha)^{-1}\rangle$$

$$= (\mathbb{Z}\langle c\rangle) * (\mathbb{Z}\langle a\rangle \oplus \mathbb{Z}\langle b\rangle) / \langle c^2 a^{-1}\rangle$$

$$= \langle a, b, c : aba^{-1}b^{-1}, c^2a^{-1}\rangle \quad \leadsto a = c^2$$

$$= \langle b, c : c^2bc^{-2}b^{-1}\rangle$$

(b) $H_i(X)$:

Let $A = M$, $B = T^2$, $A \cap B = S^1$, $A \cup B = X$ as before

Now we have:

$$H_i(A) \cong H_i(M) = \begin{cases} \mathbb{Z} & i = 0,1 \\ 0 & i > 1 \end{cases}$$

$$H_i(B) \cong H_i(T^2) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & i = 1 \\ \mathbb{Z} & i = 2 \\ 0 & i > 2 \end{cases}$$

$$H_i(A \cap B) \cong H_i(S^1) = \begin{cases} \mathbb{Z} & i = 0,1 \\ 0 & i > 1 \end{cases}$$

Now apply Mayer-Vietoris:

$$0 \to H_2(A) \oplus H_2(B) \to H_2(X) \to H_1(A \cap B) \to H_1(A) \oplus H_1(B) \to H_1(X)$$
$$\to H_0(A \cap B) \to H_0(A) \oplus H_0(B) \to H_0(X) \to 0$$

$$\Rightarrow \quad 0 \to \mathbb{Z} \to H_2(X) \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \to H_1(X) \to \mathbb{Z} \to$$
$$\mathbb{Z} \oplus \mathbb{Z} \to H_0(X) \to 0.$$

Now, X still path-connected, hence $\underline{H_0(X) \cong \mathbb{Z}}$

and $H_1(X) = \pi_1(X) / \pi_1(X)' = \langle \gamma, \beta : \gamma^2 \beta \gamma^{-2} \beta^{-1} \rangle / \langle \gamma\beta\gamma^{-1}\beta^{-1} \rangle$

$= \langle \gamma, \beta : \gamma\beta\gamma^{-1}\beta^{-1} \rangle = \underline{\underline{\mathbb{Z} \oplus \mathbb{Z} = H_1(X)}}$

So we have the sequence:

$$0 \to \mathbb{Z} \to H_2(X) \xrightarrow{\varphi} \mathbb{Z} \xrightarrow{\varepsilon} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\delta} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\gamma} \mathbb{Z} \xrightarrow{\beta} \mathbb{Z} \oplus \mathbb{Z}$$
$$\xrightarrow{\alpha} \mathbb{Z} \to 0.$$

α surjective $\Rightarrow \ker\alpha = \mathbb{Z} \Rightarrow \operatorname{im}\beta = \mathbb{Z} \Rightarrow \ker\beta = 0 \Rightarrow \operatorname{im}\gamma = 0$

$\Rightarrow \ker\gamma = \mathbb{Z} \oplus \mathbb{Z} \Rightarrow \operatorname{im}\delta = \mathbb{Z} \oplus \mathbb{Z} \Rightarrow \ker\delta = \mathbb{Z} \Rightarrow \operatorname{im}\varepsilon = \mathbb{Z} \Rightarrow \ker\varepsilon = 0$

$\Rightarrow \operatorname{im}\varphi = 0 \Rightarrow \ker\varphi = H_2(X)$.

But $0 \to \mathbb{Z} \to H_2(X)$ implies injectivity by exactness, hence

$\ker\varphi = \mathbb{Z}$, hence $\underline{\underline{H_2(X) \cong \mathbb{Z}}}$

4.) Find $\int_{\Sigma} K\,dA$ where Σ cpt, ornt, genus 2004, $\subseteq \mathbb{R}^3$

Since Σ cpt in $\mathbb{R}^3$, it is closed & bdd, hence no boundary, and we may apply Gauss-Bonnet:

$$\int_M K\,dA = 2\pi\chi(M)$$

So then:

$$\int_{\Sigma} K\,dA = 2\pi\chi(\Sigma) = 2\pi(2-2g)$$

$$= 2\pi(2-2(2004))$$

$$= 2\pi(2-4008)$$

$$= -2\pi(4006)$$

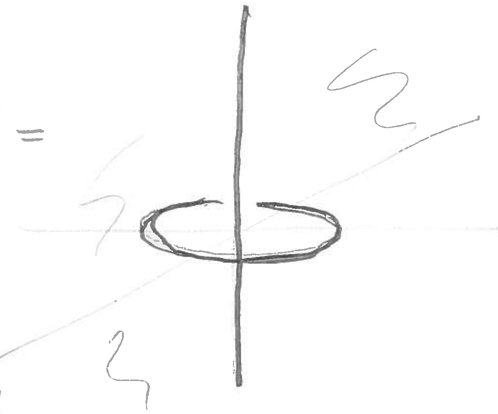
$$= \boxed{-8012\pi}$$

5. $X = \mathbb{R}^3 \setminus (A_1 \cup A_2)$ where $A_1 = \{x=y=0\} = z$ axis

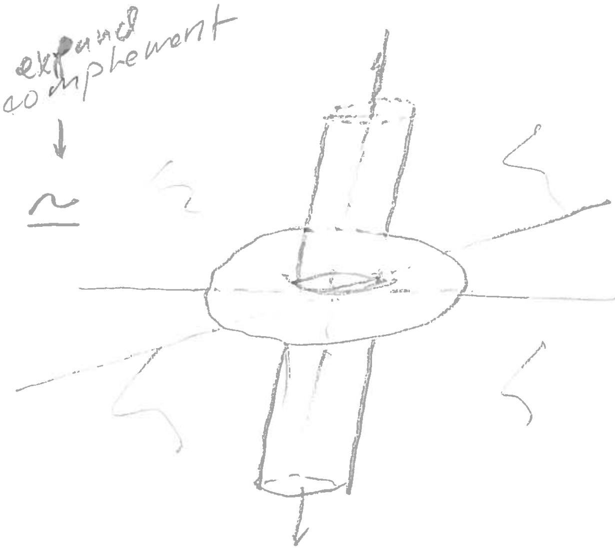
$A_2 = \{x^2+y^2=1, z=0\} \simeq S^1$

$\pi_1(X)$?

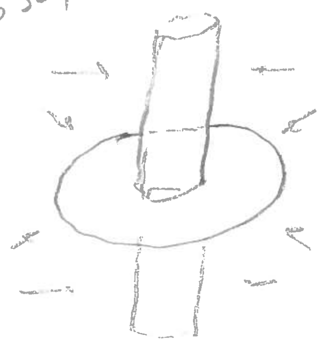
See that $X =$



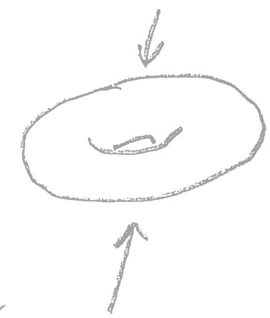
$\simeq$



$\simeq$



$\simeq$



$= T^2$

Hence $\pi_1(X) \cong \pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$.

(6.) $\nexists$ smooth $f: S^2 \to S^3$ surjective

Suppose such a map exists.
Then consider $T_pf: T_pS^2 \to T_pS^3 \cong T_pf: \mathbb{R}^2 \to \mathbb{R}^3$
which is never surjective since linear and
$\dim(\mathrm{dom} f) < \dim(\mathrm{codom} f)$, hence f has no
regular pts, hence no reg. values, hence
by Sard, $\lambda(\mathrm{im} f) = 0$

Thus $\mathrm{im} f \neq S^3$ since $\lambda(S^3) > 0$.

Hence f not surjective.

(7) f homogeneous polynomial in k real variables,
i.e. $f(tx_1, \dots, tx_k) = t^m f(x_1, \dots, x_k)$. for all $t \in \mathbb{R}$.
Prove that $M = \{f(X) = a\}$ is a $(k-1)$-dim'l submfld of $\mathbb{R}^k$
provided $a \neq 0$.
[Hint: Euler identity: $\sum_{i=1}^{k} x_i \frac{\partial f}{\partial x_i} = m \cdot f$]

Clearly $M = f^{-1}(a)$, hence apply regular value thm.:
$p \in f^{-1}(a)$, $T_p f : T_p\mathbb{R}^k \longrightarrow T_p\mathbb{R}$

$$T_p f = \left[\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_k}\right]\Big|_p$$

So then, letting $p = (p_1, \dots, p_k)$ (coord. rep'n):

$$T_p f(p) = \sum p_i \frac{\partial f}{\partial x_i}(p) = m \cdot f(p) = ma \neq 0$$

since $a \neq 0$,

hence $T_p f \neq 0$, hence surjective
since linear & codomain $= \mathbb{R}$.

Hence $f^{-1}(a) = M$ a submanifold
of dim $\dim \mathbb{R}^k - \dim \mathbb{R} = \underline{k-1}$.