Geometry/Topology Qualifying Exam

September 2004

Solve all **SEVEN** *problems. Partial credit will be given to partial solutions.*

1. Prove that a k-form ω on a k-dimensional torus T^k is exact if and only if $\int_{T^k} \omega = 0$.

2. Consider the following $(n-1)$-form ω on $\mathbf{R}^n$ with coordinates $(x_1, \ldots, x_n)$:

$$\omega = \frac{\sum_{i=1}^n (-1)^{i+1} x_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \cdots \wedge dx_n}{(\sum_{i=1}^n x_i^2)^{n/2}},$$

where $\widehat{dx_i}$ means the dx_i term is omitted.
(a) Show that the form ω is closed on $\mathbf{R}^n - \{0\}$.
(b) Compute $\int_E \omega$, where E is the ellipsoid

$$E = \left\{ \frac{x_1^2}{9} + \sum_{i=2}^n x_i^2 = 2004 \right\},$$

and the orientation of E is the outward orientation (induced from the compact region of $\mathbf{R}^n$ bounded by E). You may leave your answer in terms of the volume $vol(B^n)$ of the n-dimensional unit ball B^n.

3. Let X be the topological space obtained from a torus $S^1 \times S^1$ by attaching a Möbius band via a homeomorphism from the boundary circle of the Möbius band to the circle $S^1 \times \{x_0\}$ on the torus. [Here a *Möbius band* is obtained from $[0,1] \times [0,1]$ by identifying $(x, 0) \sim (1-x, 1)$ for all $x \in [0,1]$.]
(a) Compute its fundamental group $\pi_1(X)$.
(b) Compute its homology groups $H_n(X; \mathbf{Z})$ for all $n \geq 0$.

4. Carefully state the Gauss-Bonnet Theorem and use it compute the total Gaussian curvature $\int_\Sigma \kappa$, where Σ is a compact oriented surface of genus 2004 which is embedded in $\mathbf{R}^3$.

5. Let X be the topological space obtained from $\mathbf{R}^3$ (with standard coordinates (x, y, z)) by removing two subsets $A_1 = \{x = y = 0\}$ (the z-axis) and $A_2 = \{x^2+y^2 = 1, z = 0\}$ (the boundary of the unit disk in $\mathbf{R}^2 \subset \mathbf{R}^3$). Calculate the fundamental group of X.

6. Show that there exists no smooth (C^∞-differentiable) surjective map from S^2 to S^3.

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7. Let f be a homogeneous polynomial in k (real) variables. Homogeneity means that there is some positive integer m for which

$$f(tx_1, \ldots, tx_k) = t^m f(x_1, \ldots, x_k),$$

for all $t \in \mathbf{R}$ and $x_1, \ldots, x_k \in \mathbf{R}$. Prove that the set of points $x \in \mathbf{R}^k$ for which $f(x) = a$ is a $(k-1)$-dimensional submanifold of $\mathbf{R}^k$, provided $a \neq 0$. [Hint: Use Euler's identity for homogeneous polynomials, which states that $\sum_{i=1}^{k} x_i \frac{\partial f}{\partial x_i} = m \cdot f$.]