## Geometry/Topology Qualifying Exam
Fall 2003

0

1. Let $T^n$ be the $n$-dimensional torus $S^1 \times S^1 \times \cdots \times S^1$. Construct a differentiable embedding of $T^n$ in $\mathbf{R}^{n+1}$.

check

2. Let $S^n$ denote the $n$-dimensional sphere, and consider a differentiable map $f : S^n \to \mathbf{R}^n$ such that $f(S^n)$ has non-empty interior in $\mathbf{R}^n$.
   a) Warm-up: Show there there is at least one point $x \in S^n$ where $f$ is a local diffeomorphism, namely such that there exists an open neighborhood $U \subset M$ of $x$ such that restriction $f_{|U} : U \to f(U)$ is a diffeomorphism.
   b) Show that there exists at least two points $x, y \in S^n$ such that $f$ is a local diffeomorphism at $x$ and $y$, $f$ is orientation-preserving at $x$, and $f$ is orientation-reversing at $y$.

3. Let $M$ be a manifold with fundamental group isomorphic to $(\mathbb{Z}/2) \times (\mathbb{Z}/3) \times (\mathbb{Z}/5)$. Up to isomorphism, how many 3-fold covers does it have? Recall that a 3-fold cover is a covering map $p : \widetilde{M} \to M$ such that each $p^{-1}(x)$ consists of 3 points, and that two such covers $p : \widetilde{M} \to M$ and $p' : \widetilde{M}' \to M$ are isomorphic if there exists a homeomorphism $\varphi : \widetilde{M} \to \widetilde{M}'$ such that $p' \circ \varphi = p$.

4. Let $M$ be a manifold of dimension $n$, and let $\omega$ be a differential form of degree $n-1$ on $M$. Suppose that $\int_N \omega = 0$ for every $(n-1)$-dimensional submanifold $N$ of $M$. Show that $d\omega = 0$. (hint: look at small spheres.)

5. Let $S^n$ denote the $n$-dimensional sphere and define $X = S^1 \times S^2$. Also, choose a point $p_n \in S^n$, for $n = 1, 2, 3$, and take the quotient $Y$ of the disjoint union of $S^1$, $S^2$, $S^3$ by the equivalence relation identifying $p_1$, $p_2$, $p_3$ to a single point $p \in Y$.
   a) Calculate the homology groups of $X$ and of $Y$.
   b) Calculate the fundamental groups as well.
   c) Are these spaces homeomorphic?

6. Let $T = S^1 \times S^1$ denote the 2-dimensional torus. Identify the circle $S^1$ to $\{z \in \mathbb{C}; |z| = 1\}$, and the 2-dimensional disk $B^2$ to $\{z \in \mathbb{C}; |z| \leq 1\}$ in the complex plane $\mathbb{C}$. Adjoin to $T$ two copies $D_1$ and $D_2$ of $B^2$, where the boundary $\partial D_1 = \partial B^2$ of the disk $D_1$ is glued to $S^1 \times \{1\}$ by the map $z \mapsto z^3$ and where the boundary $\partial D_2$ of $D_1$ is glued to $\{1\} \times S^1$ by the map $z \mapsto z^5$. Calculate the fundamental group of $X$.

1. Use polar coordinates and induction

2. a) Sard, b) $\deg f = 0$ because $f$ is not surj ($S^n$ compact, $\mathbb{R}^n$ is not), Sard and definition of degree.

3. Classification of $n$-sheeted covering spaces

4. Stokes, definition of integration using charts.

5. a) For $X$, Mayer Vietoris, For $Y$ $\tilde{H}_n(\vee_\alpha X_\alpha) \cong \oplus_\alpha \tilde{H}_n(X_\alpha)$ provided $(X_\alpha, x_\alpha)$ are good pairs.
   b) $\pi_1(A \times B) = \pi_1(A) \times \pi_1(B)$ if $A$ and $B$ are path connected.
   c) Higher homotopy groups.

6. Van Kampen

(2) $f: S^n \to \mathbb{R}^n$ s.t. $f(S^n)^\circ \neq \emptyset$.

(a) Show $\exists\, p \in S^n$ s.t. $f$ is a local diffeomorphism.

$S^n$ cpt, hence $f(S^n)$ cpt $\Rightarrow$ closed and bdd, and $f(S^n)^\circ \neq \emptyset$, hence $\lambda(f(S^n)) > 0$.
Now by Sard's theorem, the regular values will have full measure in $f(S^n)$, hence we may select a regular value $x \in f(S^n)$, i.e. we may select a regular point $p \in f^{-1}(x) \neq \emptyset$.

Since $p$ regular, $T_pf: T_pS^n \to T_p\mathbb{R}^n$ is surjective, i.e. $T_pf: \mathbb{R}^n \to \mathbb{R}^n$ is surjective, hence invertible by equality of dimension + linearity.

By the Inverse Fn. Thm, $\exists$ neighborhood $U$ of $p$ s.t. $f|_U: U \to f(U)$ is a diffeomorphism (i.e. $f$ local diffeo. at $p$).

(b) Show $\exists\, x, y \in S^n$ s.t. $f$ local diffeo at $x$ & $y$, and $f$ is orientation preserving @ $x$ & reversing @ $y$.

Since $f: S^n \to \mathbb{R}^n$ is a map from cpt to non-cpt spaces, $\deg f = 0$ (deg well-def since same dim, both orientable)
Therefore, for any regular value $p$, the geometric degree is 0.

$$0 = \deg_p f = \sum_{q \in f^{-1}(p)} \deg_q f \quad \text{where} \quad \deg_q f = \begin{cases} +1 & \text{if } f \text{ orint-pres at } q \\ -1 & \text{if } f \text{ orint-rev at } q. \end{cases}$$

$$= \operatorname{sign}(\det(T_q f))$$

Since we've shown that $f^{-1}(p) \neq \emptyset$,
we must have some $x \in f^{-1}(p)$ with $\deg_x f = +1$
and some $y \in f^{-1}(p)$ with $\deg_y f = -1$ to make the
sum $\deg_p f = 0$.

These are both regular points, hence the maps
$T_x f : \mathbb{R}^n \to \mathbb{R}^n$ and $T_y f : \mathbb{R}^n \to \mathbb{R}^n$ are both surjective,
hence invertible, and $\det T_x f > 0$ & $\det T_y f < 0$

So $f$ is a local diffeomorphism at $x$ that is orientation preserving
& $f$ is a local diffeomorphism at $y$ that is orientation-reversing

3.

(3.) $M$ mfld with $\pi_1(M) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$.

Up to iso, how many 3-fold covers of $M$?

Recall that 3-sheeted covering spaces of $M$ are in correspondence with the homomorphisms $f: \pi_1(M) \to S_3$, so consider these, i.e.

$$f: \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \to S_3$$

Since the domain is abelian $f$ may only map to commuting permutations; $f$ is determined by its action on the generators, so let $\langle\alpha\rangle = \mathbb{Z}_2$, $\langle\beta\rangle = \mathbb{Z}_3$, $\langle\gamma\rangle = \mathbb{Z}_5$

Then $f(\alpha)$ is order 2
$f(\beta)$ is order 3
$f(\gamma)$ is order 5

Recall $S_3 = \{id, (123), (123)^2 = (132), (12), (23), (13)\}$

- $5 \nmid 3! = 6 = |S_3|$, hence there are no order 5 elements, hence $f(\gamma) = id$ for all hom.'s $f$.
- $f(\alpha)$ is an order 2 permutation; there are $id, (12), (23), (13)$
- $f(\beta)$ is an order 3 permutation, there are $id, (123), (132)$

Now all of $f(\alpha), f(\beta), f(\gamma)$ must commute, but no nontrivial transposition will commute with $(123)$ or $(132)$, and vice-versa.

So we get

| $f(\alpha)$ | $f(\beta)$ | $f(\gamma)$ |
|---|---|---|
| id | id | id |
| (12) | id | id |
| (23) | id | id |
| (13) | id | id |
| id | (123) | id |
| id | (132) | id |

hence there are 6 total.

(4.) $M$ mfld dim $n$, $\omega \in \Omega^{n-1}(M)$, $\int_N \omega = 0$ for every $(n-1)$-dimensional submfld $N \subseteq M$. 4.

Show $d\omega \equiv 0$:

Consider the point $p \in M$ and coordinate neighborhood $(U, \varphi)$ of $p$. Then consider the point $\varphi(p) \in \mathbb{R}^n$ and consider the $\varepsilon$-ball centered there, $B^n_\varepsilon(\varphi(p))$, and then consider the $p$-nbhd $\varphi^{-1}(B^n_\varepsilon(\varphi(p)))$.

$$\int_{\varphi^{-1}(B^n_\varepsilon(\varphi(p)))} d\omega = \int_{B^n_\varepsilon(\varphi(p))} (\varphi^{-1})^*(d\omega) = \int_{B^n_\varepsilon(\varphi(p))} d(\varphi^{-1})^*(\omega) \overset{\text{Stokes}}{=} \int_{S^{n-1}_\varepsilon(\varphi(p))} (\varphi^{-1})^*(\omega)$$

$$= \int_{\varphi^{-1}(S^{n-1}_\varepsilon(\varphi(p)))} d\omega = 0$$ since $\varphi$ homeom. onto image, $\varphi^{-1}(S^{n-1}_\varepsilon(\varphi(p)))$ is $(n-1)$-submfld of $M$.

This is true for any $\varepsilon > 0$, hence $d\omega_p = 0$; but $p$ was arbitrary, so $d\omega \equiv 0$

(5.) $X = S^1 \times S^2$, $Y = S^1 \vee S^2 \vee S^3$

(a) $H_i(X)$, $H_i(Y)$

(b) $\pi_1(X)$, $\pi_1(Y)$

(c) $X \simeq Y$ ? :

---

(a) Homology of $X$ : Let $A = S^2 \times ($ , $B = S^2 \times )$

$A \cap B = S^2 \sqcup S^2$, $A \cup B = S^2 \times S^1 = X$

$X$ is still path connected, hence $H_0(X) \cong \mathbb{Z}$; now apply Mayer-Vietoris

$H_3(S^2 \sqcup S^2) \to H_3(S^2) \oplus H_3(S^2) \to H_3(X) \to H_2(S^2 \sqcup S^2) \to H_2(S^2) \oplus H_2(S^2)$

$\to H_2(X) \to H_1(S^2 \sqcup S^2) \to H_1(S^2) \oplus H_1(S^2) \to H_1(X) \to H_0(S^2 \sqcup S^2)$

$\to H_0(S^2) \oplus H_0(S^2) \to H_0(X) \to 0$

$\Rightarrow 0 \to \underbrace{H_3(X) \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to H_2(X)}_{(*)} \to 0 \to 0 \to H_1(X)$

$\to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to H_0(X) \cong \mathbb{Z} \to 0$

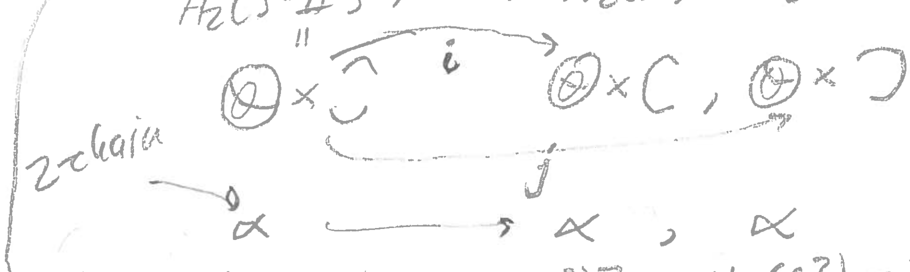
Consider portion $(*)$ :

$$0 \to H_3(X) \xrightarrow{\partial_3} H_2(S^2 \sqcup S^2) \xrightarrow{(i^*, j^*)} H_2(S^2) \oplus H_2(S^2) \xrightarrow{q} H_2(X) \to 0$$

$$\mathbb{Z} \oplus \mathbb{Z} \cong H_2(S^2) \oplus H_2(S^2) \cong H_2(S^2 \sqcup S^2), \qquad H_2(S^2) \oplus H_2(S^2) \cong \mathbb{Z} \oplus \mathbb{Z}$$

Now consider the map $(i^*, j^*)$: ; it sends $(i^*, j^*)(\alpha) = (\alpha, \alpha)$

$$H_2(S^2 \sqcup S^2) \xrightarrow{(i^*, j^*)} H_2(S^2) \oplus H_2(S^2)$$




so $\operatorname{im}(i^*, j^*) = \{(\alpha, \alpha) : \alpha \in H_2(S^2)\} = H_2(S^2) = \mathbb{Z}$,

hence $\ker(i^*, j^*) = \mathbb{Z}$.

Now, by exactness, $\partial_3$ is injective and $q$ is surjective, so:

$$H_3(X) = \operatorname{im}\partial_3 = \ker(i^*, j^*) = \mathbb{Z}$$

and

$$\mathbb{Z} = \operatorname{im}(i^*, j^*) = \ker q \Rightarrow \operatorname{im} q = \mathbb{Z} \quad \text{since domain} = \mathbb{Z}^2$$

but $H_2(X) = \operatorname{im} q = \mathbb{Z}$

Finally, for the rest of the sequence, we have:

$$0 \to H_1(X) \xrightarrow{\alpha} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\beta} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\gamma} \mathbb{Z} \to 0$$

Then by exactness $\alpha$ is injective and $\gamma$ surjective.

So $H_1(X) = \operatorname{im}\alpha = \ker\beta$

But $\operatorname{im}\gamma = \mathbb{Z} \Rightarrow \ker\gamma = \mathbb{Z} \Rightarrow \operatorname{im}\beta = \mathbb{Z} \Rightarrow \ker\beta = \mathbb{Z}$,

hence $H_1(X) \cong \mathbb{Z}$.

So: $H_i(X) \cong \begin{cases} \mathbb{Z} & i=0,1,2,3 \\ 0 & i>3 \end{cases}$

Homology of $Y$. Each sphere is a good pair with the basepoint of the wedge, hence:

$$\tilde{H}_i(S^1 \vee S^2 \vee S^3) \cong \tilde{H}_i(S^1) \oplus \tilde{H}_i(S^2) \oplus \tilde{H}_i(S^3) \cong \begin{cases} 0 & i=0 \\ \mathbb{Z} & i=1 \\ \mathbb{Z} & i=2 \\ \mathbb{Z} & i=3 \end{cases}$$

Recall $H_i(S^1\vee S^2\vee S^3) \cong \tilde{H}_i(S^1\vee S^2\vee S^3)$ for $i>0$

& $H_0(S^1\vee S^2\vee S^3) \cong \tilde{H}_0(S^1\vee S^2\vee S^3) \oplus \mathbb{Z} \cong \mathbb{Z}$

hence $H_i(S^1\vee S^2\vee S^3) \cong \begin{cases} \mathbb{Z} & i=0,1,2,3 \\ 0 & i>3 \end{cases}$

(b). $\pi_1(X), \pi_1(Y)$ ?

$\pi_1(X) = \pi_1(S^1 \times S^2) \cong \pi_1(S^1) \times \pi_1(S^2) = \mathbb{Z} \times 1 = \mathbb{Z}$

$\pi_1(Y) = \pi_1(S^1 \vee S^2 \vee S^3) \cong \pi_1(S^1) * \pi_1(S^2) * \pi_1(S^3) = \mathbb{Z}$

Seifert van Kampen

(c) Are X & Y homeomorphic?

#1 No. Consider the higher homotopy groups. First:

$\pi_2(X) = \pi_2(S^1 \times S^2) \cong \pi_2(S^1) \times \pi_2(S^2) = 0 \times \mathbb{Z} = \mathbb{Z}$

Now consider the covering space for $S^1 \vee S^2 \vee S^3$.


It has infinitely many copies of $S^2$, hence $\pi_2(S^1 \vee S^2 \vee S^3)$ will be infinitely generated, hence $\pi_2(X) \not\cong \pi_2(Y)$.

#2 No. If you remove a pt. from $S^1 \times S^2$ it is still connected ($S^1 \times S^2$ is manifold), but remove wedge point of $S^1 \vee S^2 \vee S^3$ will yield a ~~dis~~connected space (i.e. wedge is not mfld), hence not homeomorphic!

(6) $T^2 = S^1 \times S^1$, $D_1 = D_2 = B^2$

Let $X = T^2 \amalg D_1 \amalg D_2 / {\partial D_1 \sim_1 S^1 \times \{1\} \atop \partial D_2 \sim_2 \{1\} \times S^1}$ where $\partial D_1 \longrightarrow S^1$, $z \longmapsto z^3$; $\partial D_2 \longrightarrow S^1$, $z \longmapsto z^5$

Calculate $\pi_1(X)$ :


Let $A_1 = D_1$, $B_1 = T^2$

Then $A_1 \cap B_1 = \partial D_1 \cong S^1$

We'll consider $X_1 = T^2 \amalg D_1 / \sim_1$ :

Now consider the inclusions :

$i_{A_1} : A_1 \cap B_1 \longrightarrow A_1$ , $i_{B_1} : A_1 \cap B_1 \longrightarrow B_1$ $\quad \alpha = b^3$


$i^*_{A_1} : \pi_1(A_1 \cap B_1) \longrightarrow \pi_1(A_1)$, $\pi_1(A_1 \cap B_1) \cong \mathbb{Z}\langle\alpha\rangle$, $\pi_1(A_1) \cong 1$ ,

$i^*_{B_1} : \pi_1(A_1 \cap B_1) \longrightarrow \pi_1(B_1) \cong \mathbb{Z}\langle b\rangle \oplus \mathbb{Z}\langle c\rangle$, $\pi_1(A_1 \cap B_1) \cong \mathbb{Z}\langle\alpha\rangle$, $\alpha \longmapsto b^3$

So $i^*_{A_1}(\alpha) = 1$ + $i^*_{B_1}(\alpha) = b^3$

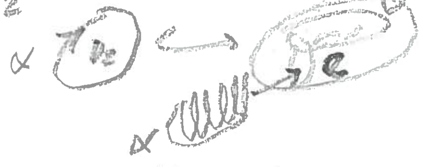
Then $\pi_1(X_1) \cong \pi_1(A) * \pi_1(B_1) / \langle i^*_{A_1}(\alpha)\, i^*_{B_1}(\alpha)^{-1} \rangle \cong \langle b\rangle \oplus \langle c\rangle / \langle b^{-3}\rangle \cong \mathbb{Z}_3 \oplus \mathbb{Z}\langle c\rangle$

Now we'll attach $D_2$ to $X_1$, i.e.

Let $A_2 = D_2$ and $B_2 = X_1$ ; again $A_2 \cap B_2 = \partial D_2 \cong S^1$.

Then consider the inclusions :

$i_{B_2} : A_1 \cap B_1 \hookrightarrow B_1$ $\rightsquigarrow$ $i^*_{B_2} : \pi_1(A_1 \cap B_1) \longrightarrow \pi_1(B_1)$, $\pi_1(A_1 \cap B_1) = \mathbb{Z}\langle\alpha\rangle$, $\pi_1(B_1) \cong \mathbb{Z}\langle b\rangle \oplus \mathbb{Z}\langle c\rangle$




$\longrightarrow$ $\underline{i^*_{B_2}(\alpha) = c^5}$

Now, $A_2 = D_2$ is contractible, so $i_{A_2}^*(\alpha) = 1$,

So we have now:

$$\pi_1(X) \cong \pi_1(X_1) * \pi_1(A_2) / \langle c^5 \rangle$$

$$\cong (\mathbb{Z}_3 \oplus \mathbb{Z}\langle c \rangle * 1) / \langle c^5 \rangle \cong \boxed{\mathbb{Z}_3 \oplus \mathbb{Z}_5}$$