Geometry/Topology Qualifying Exam

Fall 2003

1. Let T^n be the n–dimensional torus $S^1 \times S^1 \times \cdots \times S^1$. Construct a differentiable embedding of T^n in $\mathbf{R}^{n+1}$.

2. Let S^n denote the n–dimensional sphere, and consider a differentiable map $f : S^n \to \mathbf{R}^n$ such that $f(S^n)$ has non-empty interior in $\mathbf{R}^n$.

a) Warm-up: Show there there is at least one point $x \in S^n$ where f is a local diffeomorphism, namely such that there exists an open neighborhood $U \subset M$ of x such that restriction $f_{|U} : U \to f(U)$ is a diffeomorphism. .

b) Show that there exists at least two points x, $y \in S^n$ such that f is a local diffeomorphism at x and y, f is orientation-preserving at x, and f is orientation-reversing at y.

3. Let M be a manifold with fundamental group isomorphic to $(\mathbb{Z}/2) \times (\mathbb{Z}/3) \times (\mathbb{Z}/5)$. Up to isomorphism, how many 3–fold covers does it have? Recall that a 3–fold cover is a covering map $p : \widetilde{M} \to M$ such that each $p^{-1}(x)$ consists of 3 points, and that two such covers $p : \widetilde{M} \to M$ and $p' : \widetilde{M}' \to M$ are isomorphic if there exists a homeomorphism $\varphi : \widetilde{M} \to \widetilde{M}'$ such that $p' \circ \varphi = p$.

4. Let M be a manifold of dimension n, and let ω be a differential form of degree $n-1$ on M. Suppose that $\int_N \omega = 0$ for every $(n-1)$–dimensional submanifold N of M. Show that $d\omega = 0$. (hint: look at small spheres.)

5. Let S^n denote the n-dimensional sphere and define $X = S^1 \times S^2$. Also, choose a point $p_n \in S^n$, for $n = 1, 2, 3$, and take the quotient Y of the disjoint union of S^1, S^2, S^3 by the equivalence relation identifying p_1, p_2, p_3 to a single point $p \in Y$.

a) Calculate the homology groups of X and of Y.

b) Calculate the fundamental groups as well.

c) Are these spaces homeomorphic?

6. Let $T = S^1 \times S^1$ denote the 2-dimensional torus. Identify the circle S^1 to $\{z \in \mathbb{C}; |z| = 1\}$, and the 2–dimensional disk B^2 to $\{z \in \mathbb{C}; |z| \leq 1\}$ in the complex plane $\mathbb{C}$. Adjoin to T two copies D_1 and D_2 of B^2, where the boundary $\partial D_1 = \partial B^2$ of the disk D_1 is glued to $S^1 \times \{1\}$ by the map $z \mapsto z^3$ and where the boundary ∂D_2 of D_1 is glued to $\{1\} \times S^1$ by the map $z \mapsto z^5$. Calculate the fundamental group of X.