Qualifying Exam in Geometry/Topology Fall 2000

1. Let ω be a 1-form defined on the sphere $S^2 = \{x \in R^3 | |x| = 1\}$. Assume ω is invariant under rotations, i.e. $\phi^*\omega = \omega$ for any $\phi \in SO(3)$, show $\omega = 0$.

2. Show the set $M = \{x \in R^4 | x_1 x_2 = x_3 x_4, |x| = 1\}$ is a smooth orientable surface.

3. Let M, N be smooth manifolds of dimension n, and $\pi : M \to N$ be a smooth map which is onto and has rank n at each point. Prove or disprove the statements:

   (a) π is locally a diffeomorphism;

   (b) π is a covering map. → see other test regarding this too

4. Let S^1 be the unit circle in $R^2 = R^2 \times \{0\} \subset R^3$. Compute the fundamental group of $R^3 - S^1$.

5. Compute the homology of $R^3 - S^1$ with coefficients in Z.

6. Let $f : RP^2 \to T^2$ be a continuous map from the projective plane RP^2 to the torus $T^2 = S^1 \times S^1$.

   (a) Show that the induced homomorphism $f_* : \pi_1(RP^2) \to \pi_1(T^2)$ is trivial.

   (b) Show that f is homotopic to a constant map.

1. $\forall\, x \in S^2$, $v \in T_x S^2$, $\exists\, \varphi \in SO(3)$ s.t. $\varphi(x) = x$ and $\varphi(v) = -v$. Then $\omega_x v = \omega_x(-v) \Rightarrow 2\omega_x(v) = 0 \Rightarrow \omega_x(v) = 0\ \forall x, \forall v$.

2. Regular Value Theorem

3. a) Tangent map is an isomorphism $\Rightarrow \pi$ is a local diffeomorphism.

   b) Counter example: $\pi : \mathbb{R} \to S^1$ $\pi(t) = t^2$.

4. $\mathbb{R}^3 - S^1 \overset{h.e.}{\cong} S^1 \vee S^2$

5. $\tilde{H}_n(\vee_\alpha X_\alpha) \cong \bigoplus_\alpha \tilde{H}_n(X_\alpha)$ provided (X_α, x_α) are good pairs, ie, x_α is a def. retract of some neighborhood in X_α.

6. a) Any homomorphism $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$ is trivial

   b) Lifting Criterion

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(1.) $\omega \in \Omega^1(S^2)$, $\phi^*\omega = \omega$ for all $\phi \in SO_3 \Rightarrow \omega = 0$

- Recall that SO_3 acts transitively on TS^2, i.e. the action has one orbit, hence for any two $(p,v), (q,w) \in TS^2$, $\exists\, \phi \in SO_3$ such that $\phi^*(p,v) = (\phi(p), T_p\phi(v)) = (q,w)$.
- In particular, let $\phi_0 \in SO_3$ be the isometry s.t. $\phi_0^*(p,v) = (p,-v)$; Then:

$$\omega_p(v) = \phi_0^*(\omega_p(v)) = \omega_{\phi_0(p)}(T_p\phi_0(v))$$
$$= \omega_p(-v)$$

$$\Rightarrow \omega_p(v) = -\omega_p(v) \Rightarrow \omega_p(v) = 0$$

But p, v arbitrary, hence $\omega \equiv 0$.

(2.) Show $M = \{x \in \mathbb{R}^4 : x_1x_2 = x_3x_4,\ |x| = 1\}$ smooth, ont surface

Consider the map $f: S^3 \longrightarrow \mathbb{R}$
$$(x_1,x_2,x_3,x_4) \mapsto x_1x_2 - x_3x_4.$$

Then clearly $f^{-1}(0) = M \subseteq S^3$ (by the $|x|=1$ condition)

Now for $p \in f^{-1}(0)$, consider $T_pf: T_pS^3 \to T_0\mathbb{R}$

$$T_pf = [x_2\ x_1\ -x_4\ -x_3]\big|_p$$

$p \in S^3$, hence not all $x_i = 0$ since $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$,

therefore $T_pf \neq 0$, hence surjective by linearity.

So $M \subseteq S^3 \subseteq \mathbb{R}^4$ submanifold.

(3.) M, N manifolds, dim n, $\pi: M \to N$ onto, rank n at every point. T/F:

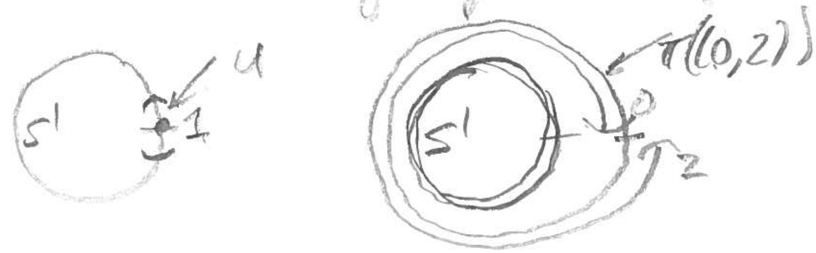
(a) π is locally a diffeomorphism:

Since $T_p\pi : T_pM \to T_{\pi(p)}N$ is rank n for all $p \in M$ and $T_pM \cong \mathbb{R}^n$, $T_{\pi(p)} \cong \mathbb{R}^n$, $T_p\pi$ is invertible by linearity. Hence π a local diffeo. at p by inverse fn. theorem, hence π a local diffeo. at all points, hence π local diffeo. (TRUE)

(b) π covering map?

FALSE; consider the map $\pi : (0,2) \to S^1$
$t \mapsto e^{2\pi i t}$

Clearly π is surjective and a local diffeomorphism but consider the inverse image of a neighborhood of 1 in S^1:



The inverse image $f^{-1}(U) = (0,\varepsilon) \cup (1-\varepsilon, 1+\varepsilon) \cup (2-\varepsilon, 2)$ but clearly $(0,\varepsilon)$ & $(2-\varepsilon, 2)$ do not map homeomorphically onto U since $f(0,\varepsilon)$, $f(2-\varepsilon,2)$ do not contain 1,

hence no nbhd of 1 is evenly covered, hence π not covering map.

④ Compute $\pi_1(\mathbb{R}^3 \setminus S^1)$:

$\mathbb{R}^3\setminus S^1 \simeq B^3 \setminus S^1 \simeq$  $\simeq$  $\simeq$ 

$= S^2 \vee S^1$

So $\pi_1(\mathbb{R}^3\setminus S^1) \cong \pi_1(S^2 \vee S^1)$

$\cong \pi_1(S^2) * \pi_1(S^1) \cong \mathbb{Z}$.

⑤ Compute $H_i(\mathbb{R}^3\setminus S^1)$:

Since $\mathbb{R}^3\setminus S^1 \simeq S^2\vee S^1$, we have:

$\tilde{H}_i(\mathbb{R}^3\setminus S^1) \cong \tilde{H}_i(S^2\vee S^1) \cong \tilde{H}_i(S^2)\oplus\tilde{H}_i(S^1)$ since the spheres are both a good pair with base point.

Hence $\tilde{H}_i(\mathbb{R}^3\setminus S^1) \cong \begin{cases} 0 & i=0 \\ \mathbb{Z} & i=1 \\ \mathbb{Z} & i=2 \\ 0 & i>2 \end{cases} \Rightarrow \boxed{H_i(\mathbb{R}^3\setminus S^1) = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z} & i=1 \\ \mathbb{Z} & i=2 \\ 0 & i>2 \end{cases}}$

⑥ $f: \mathbb{R}P^2 \to T^2$

(a) $f_*: \pi_1(\mathbb{R}P^2) \to \pi_1(T^2)$ is trivial

(b) $f \simeq$ const.

(a) We know $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$ & $\pi_1(T^2) \cong \mathbb{Z}\oplus\mathbb{Z}$

Then the induced hom. is $f_*: \mathbb{Z}_2 \to \mathbb{Z}\oplus\mathbb{Z}$; $\mathbb{Z}\oplus\mathbb{Z}$ has no torsion elts, hence $f_*(1) = 0$, hence $f_* \equiv 0$.

(b) Consider the covering $p: \mathbb{R}^2 \to T^2$. Then since $p_*(\pi_1(\mathbb{R}^2)) = 0$ & $f_* \equiv 0$, we have $f_*(\pi_1(\mathbb{R}P^2)) \subseteq p_*(\pi_1(\mathbb{R}^2))$, hence we have lift:

$$\begin{tikzcd} & \mathbb{R}^2 \arrow[d, "p"] \\ \mathbb{R}P^2 \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & T^2 \end{tikzcd}$$

; $\tilde{f}$ map into contractible space, hence $\tilde{f} \simeq$ const by homotopy g_t; hence $f \simeq$ const. by $p\circ g_t$.