

Cohomology, and Other Results of Algebraic Topology 1

Setup: Let R be a commutative ring w/ 1.

X is a topological space.

Def/ The n^{th} cochain group $C^n(X; R)$

is defined as $\text{Hom}_R(C_n(X; R), R)$

the R -module of R -linear maps $C_n(X; R) \rightarrow R$,

where $C_n(X; R)$ is the free R -module generated by n -simplices $\sigma: \Delta^n \rightarrow X$ where

Δ^n is the standard n -simplex $\Delta^n =$

$$= \left\{ (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} : x_i \geq 0 \text{ and } x_0 + x_1 + \dots + x_n = 1 \right\}$$

Def/ The coboundary map $\delta: C^n(X; R) \rightarrow C^{n+1}(X; R)$

defined on cochains $\varphi \in C^n(X; R)$ is given by

$$\delta(\varphi)(c) = \sum_{i=0}^{n+1} (-1)^i \varphi(\partial_i c) \text{ for all } (n+1)\text{-chains } c \in C_{n+1}(X; R)$$

From Milnor, this sign seems outdated (Hatcher nor Bonahon use it)

Here, $\partial : C_{n+1}(X; R) \rightarrow C_n(X; R)$

is the boundary map defined by

$$\partial(\sigma) = \sum_{i=0}^{n+1} (-1)^i (\sigma \circ f_i) \text{ for all } (n+1)\text{-simplices } \sigma,$$

where $f_i : \Delta^n \rightarrow \Delta^{n+1}$ is the i^{th} face map.

Proposition : $\partial^2 = 0$.

Def/ Let $Z^n(X; R) := \ker \partial : C^n(X; R) \rightarrow C^{n+1}(X; R)$

and let $B^n(X; R) := \text{im } \partial : C^{n+1}(X; R) \rightarrow C^n(X; R)$

The n^{th} cohomology group $H^n(X; R)$ is

defined to be $Z^n(X; R) / B^n(X; R)$.

Notation / when the ring R is understood,
we simply write $C^n(X)$, $H^n(X)$, etc.

Theorem : Given $[c] \in H^n(X)$ and $[c] \in H_n(X)$, so $c \in Z^n(X)$ and $c \in Z^n(X)$, then the scalar $\varphi(c) \in R$ is independent of the

representatives. we therefore have a 2
well-defined map of R -modules

$$H^n(X) \longrightarrow \text{Hom}_R(H_n(X), R)$$

canonically defined by $[\varphi] \mapsto ([c] \mapsto \varphi(c))$.

The result is that this canonical map
is an isomorphism assuming R is a PID
and either $H_{n-1}(X) = 0$ or $H_{n-1}(X)$ is a free
 R -module.

Cor/ If $R = F$ is a field, then
 $H^n(X)$ is canonically isomorphic to
 $\text{Hom}_F(H_n(X), F)$ for all n .

Universal Cover and the Fundamental Group

Def/ A covering space $(Y, p: Y \rightarrow X)$ of a space X is
a universal cover of X if Y is simply connected.

Thm/ A universal cover is unique up to homeomorphism preserving the fibers.

Rmk/ we thus refer to the universal cover of X .

Thm/ Let $(Y, p: Y \rightarrow X)$ be the universal cover of a space X .
(X is assumed path connected, locally path connected, semi-locally simply connected) Let $\text{Aut } Y$

be the group $\{ \varphi: Y \rightarrow Y \text{ homeo s.t. } p \circ \varphi = p \}$.

Then $\pi_1(X) \cong \text{Aut}(Y)$.

Finite Cell Complexes (adapted from Banahan's notes)

Def/ A finite cell complex is a space X which can be written as

$$X = X^{(0)} \cup X^{(1)} \cup \dots \cup X^{(n)}$$

where

1) $X^{(0)}$ consists of finitely many points.

2) Each $X^{(i+1)}$ is obtained from $X^{(i)}$ by

gluing finitely many copies $B_1^{i+1}, B_2^{i+1}, \dots, B_{k_i}^{i+1}$ of the

$(i+1)$ -dim ball B^{i+1} along continuous maps $\varphi_j: S_j^i \rightarrow X^{(i)}$

where $S_j^i = \text{boundary of } B_j^{i+1} \cong \text{copy of } i\text{-dim sphere } S^i$

Rmks/. So each $X^{(i)}$ is contained in $X^{(i+1)}$

• $X^{(i+1)}$ is determined by maps $\varphi_j^i: S_j^i \rightarrow X^{(i)}$

such that $X^{(i+1)} = X^{(i)} \amalg B_1^{i+1} \amalg \dots \amalg B_{k_i}^{i+1} / \sim$

where for $X \in S_j^i \subset B_j^{i+1}$, $X \sim \varphi_j^i(X) \in X^{(i)}$

• $\pi_1(X^{(0)}, x_0) \cong \{e\}$. • X is of dim n is $X = X^{(n)}$ for some $n \geq 0$.

Proposition / Definition

A finite graph is a finite cell complex of dimension 1 (i.e. $X = X^{(1)}$).

For a finite graph which is connected, has v vertices (i.e. $\#(X^{(0)}) = v$),

and has e edges (i.e. $k_1 = e$), then

$\pi_1(X, x_0) \cong F^{e-v+1}$ the free group on $e-v+1$ generators, also written $\mathbb{Z}^{*(e-v+1)}$.

Cellular Homology (adapted from Hatcher)

Lemma/ Let $X = X^{(0)} \cup X^{(1)} \cup \dots \cup X^{(n)}$ be a finite cell complex. Over $\mathbb{Z}$,

$$(a) \quad H_p(X^{(k)}, X^{(k-1)}) \cong \begin{cases} 0, & p \neq k \\ \mathbb{Z}^{\# \{k\text{-cells}\}}, & p = k \end{cases} \quad (\mathbb{Z}^0 \equiv 0)$$

$$(b) \quad H_p(X^{(k)}) \cong 0 \text{ for } p > k.$$

$$(c) \quad H_p(X^{(k)}) \cong H_p(X) \text{ for } p < k.$$

Def/ Let $X = X^{(0)} \cup X^{(1)} \cup \dots \cup X^{(n)}$ be a finite cell complex. The cellular chain complex of X

is the complex

$$\dots \rightarrow H_{k+1}(X^{(k+1)}, X^{(k)}) \xrightarrow{d_{k+1}} H_k(X^{(k)}, X^{(k-1)}) \xrightarrow{d_k} H_{k-1}(X^{(k-1)}, X^{(k-2)}) \rightarrow \dots$$

where the boundary maps d_k are defined by

$$\begin{array}{c} \begin{array}{c} H_k(X^{(k-1)}) \cong 0 \\ \downarrow \partial_{k+1}^{(k+1)} \\ H_{k+1}(X^{(k+1)}, X^{(k)}) \end{array} \xrightarrow{d_{k+1}} \begin{array}{c} H_k(X^{(k)}, X^{(k-1)}) \\ \downarrow \partial_k^{(k)} \\ H_k(X^{(k)}) \end{array} \xrightarrow{d_k} \begin{array}{c} H_{k-1}(X^{(k-1)}, X^{(k-2)}) \\ \downarrow \partial_{k-1}^{(k-1)} \\ H_{k-1}(X^{(k-1)}) \end{array} \rightarrow \dots \\ \begin{array}{c} \downarrow \partial_{k+1}^{(k+1)} \\ H_{k+1}(X^{(k+1)}, X^{(k)}) \end{array} \xrightarrow{d_{k+1}} \begin{array}{c} H_k(X^{(k)}, X^{(k-1)}) \\ \downarrow \partial_k^{(k)} \\ H_k(X^{(k)}) \end{array} \xrightarrow{d_k} \begin{array}{c} H_{k-1}(X^{(k-1)}, X^{(k-2)}) \\ \downarrow \partial_{k-1}^{(k-1)} \\ H_{k-1}(X^{(k-1)}) \end{array} \rightarrow \dots \end{array}$$

Additional annotations in the diagram include: $H_k(X^{(k-1)}, X^{(k-2)}) \cong 0$, $H_k(X^{(k)}, X^{(k-1)}) \cong H_k(X)$, and $H_{k-1}(X^{(k-1)}, X^{(k-2)}) \cong 0$.

Here we are using the long exact sequence in relative homology.

$$\text{So } d_k := p_{k-1}^{(k-1)} \circ \partial_k^{(k)}.$$

It is obvious $d_k \circ d_{k+1} = 0$.

So far we have not used any of the isomorphisms marked in red on the diagram. We think of elts of $H_k(X^k, X^{k-1})$ as linear combinations of k -cells of X .

Define the n^{th} cellular homology group of X (over $\mathbb{Z}$), denoted $H_n^{\text{CW}}(X)$, as the n^{th} homology of this complex.

$$\text{Thm/ } H_n^{\text{CW}}(X) \cong H_n(X).$$

Proof (Thm): Now we use the markings in red.

$$\begin{aligned} \underline{H_k(X)} &\cong H_k(X^{k+1}) \cong \ker p_k^{(k+1)} \cong \text{im } i_k^{(k+1)} \cong H_k(X^k) / \ker i_k^{(k+1)} \\ &\cong \underline{H_k(X^k) / \text{im } \partial_{k+1}^{(k+1)}}. \end{aligned}$$

Since $p_k^{(k)}$ is injective, $\text{im } \partial_{k+1}^{(k+1)} \cong \text{im } p_k^{(k)} \circ \partial_{k+1}^{(k+1)} = \text{im } d_{k+1}$,
and $H_k(X^k) \cong \text{im } p_k^{(k)} = \ker \partial_k^{(k)}$.

Since $p_{k-1}^{(k-1)}$ is injective, $\ker d_k = \ker \partial_k^{(k)}$.

we conclude $H_k(X) \cong H_k(X^k) / \text{im } \partial_{k+1}^{(k+1)} \cong \ker d^k / \text{im } d^{k+1}$. $\square$

Rmks/ (Immediate consequences)

• we continue writing X^k instead of $X^{(k)}$.

• $H_k(X) = 0$ if X has no k -cells.

Proof:

cellular chain complex:

$$\rightarrow H_{k+1}(X^{k+1}, X^k) \xrightarrow{d_{k+1}} H_k(X^k, X^{k-1}) \xrightarrow{d_k} H_{k-1}(X^{k-1}, X^{k-2}) \rightarrow$$

$H_k(X^k, X^{k-1}) = H_k(X^{k-1}, X^{k-1})$ since $X^k = X^{k-1}$.
we are done since

$$H_k(X^{k-1}, X^{k-1}) = 0 :$$

Recall $\wedge$ $H_k(X, A) = Z_k(X, A) / B_k(X, A)$
generally

where $Z_k(X, A) = \{ C_k \in C_k(X) : \partial C_k \in C_{k-1}(A) \}$

and $B_k(X, A) = \{ C_k \in C_k(X) : \exists C_{k+1} \in C_{k+1}(X) \text{ s.t. } \partial C_{k+1} - C_k \in C_k(A) \}$

Clearly when $A=X$, $Z_k(X, X) = B_k(X, X) = C_k(X)$. $\square$

• If X has m_k k -cells, 5
then $H_n(X)$ is generated by at most
 k elements.

Proof: $H_k(X^k, X^{k-1}) \cong \mathbb{Z}^{m_k}$

is free abelian of rank m_k .

Hence $\ker d_k \subset H_k(X^k, X^{k-1})$ is

free abelian of rank $\leq m_k$

(since $\mathbb{Z}$ is a PID).

Hence $\ker d_k / \operatorname{im} d_{k+1}$ is generated

by at most m_k elts (it is not

necessarily free!) $\square$.

- If no two cells of X are in adjacent dimensions, and there are m_k k -cells,

then $H_k(X) \cong H_k(X^k, X^{k-1}) \cong \mathbb{Z}^{m_k}$.

Proof: $\rightarrow H_{k+1}(X^{k+1}, X^k) \xrightarrow{d_{k+1}} H_k(X^k, X^{k-1}) \xrightarrow{d_k} H_{k-1}(X^{k-1}, X^{k-2}) \rightarrow$

$0 \quad \quad \quad \mathbb{Z}^{m_k} \quad \quad \quad 0$

$$H_k(X) \cong \ker d_k / \operatorname{im} d_{k+1} \cong \mathbb{Z}^{m_k} / 0. \quad \square$$

example/ It can be shown $e^0 \cup e^2 \cup \dots \cup e^{2n}$ is a cellular decomposition of $\mathbb{C}P^n$.
 So $H_k(\mathbb{C}P^n) = \begin{cases} 0, & k \text{ odd} \\ \mathbb{Z}, & k \text{ even} \end{cases}$.

Goal/ Compute the boundary maps d_k in terms of degree S .

warm up/ (computation in a special case of d_1)

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$$\begin{array}{ccc} & \xrightarrow{\partial_1^{(1)}} & H_0(X^0) \\ & & \searrow p_0^{(0)} \\ H_1(X^1, X^0) & \xrightarrow{d_1} & H_0(X^0, \emptyset) \\ & & \cong H_0(X^0) \end{array}$$

Surely $p_0^{(0)} = \text{id}_{H_0(X^0)}$ so $d_1 = \partial_1^{(1)}$

the connecting homomorphism $H_1(X^1, X^0) \rightarrow H_0(X^0)$

w/ formula $[c] \mapsto [\partial c]$.

$$\begin{array}{l} c \in C_1(X^1) \\ \partial c \in C_0(X^0) \end{array}$$

If X is connected, $H_0(X) = \mathbb{Z}$.

If in addition X has a single 0-cell, then $H_0(X^0) \cong \mathbb{Z}$.

$$\text{so } \mathbb{Z} \cong H_0(X) \cong H_0(X^0) / \text{im } d_1 \cong \mathbb{Z} / \text{im } d_1.$$

so d_1 must be the 0 map.

Thm / (Cellular Boundary Formula)

Let $X = X^{(0)} \cup X^{(1)} \cup \dots \cup X^{(n)}$ be a finite cell complex, where for $n \geq k \geq 0$,

$$X^{(k+1)} = X^{(k)} \amalg_{\varphi_1^k} B_1^{k+1} \amalg \dots \amalg_{\varphi_{m_k}^k} B_{m_k}^{k+1}$$

Here φ_j^k is an attaching map $S_j^k = \partial B_j^k \rightarrow X^{(k)}$.

We may view $H_k(X^{(k)}, X^{(k-1)})$ as freely

generated over $\mathbb{Z}$ by $\{B_j^k : j=1, \dots, m_k\}$.

For $i=1, \dots, m_k$ Let $\pi_i^k : X^{(k)} \rightarrow X^{(k)}/\sim \cong S^k$ be the quotient map determined by collapsing $X^{(k)} - \text{int} B_i^k = X^{(k-1)} \amalg_{\varphi_1^k} B_1^k \amalg \dots \amalg_{\varphi_{i-1}^k} B_{i-1}^k \amalg_{\varphi_{i+1}^k} B_{i+1}^k \amalg \dots \amalg_{\varphi_{m_k}^k} B_{m_k}^k$ to a point (i.e. $x \sim y$ iff $x=y$ & $x, y \in X^{(k)} - \text{int} B_i^k$).



Then the boundary map d_k is calculated as

$$d_k(B_j^k) = \sum_{i=1}^{m_{k-1}} d_{ij} B_i^{k-1} \quad (j=1, \dots, m_k)$$

where d_{ij} is the degree of the composite map

$$S^{k-1} \xrightarrow{\sim} S_j^{k-1} \xrightarrow{\varphi_j^{k-1}} X^{(k-1)} \xrightarrow{\pi_i^{k-1}} X^{(k-1)}/\sim \cong S^{k-1}$$

Rmks/ • Recall that $H_n(S^n) \cong \mathbb{Z}$

which gives rise to the notion of degree:

If $f: S^n \rightarrow S^n$ is a continuous map,

then f induces a morphism on homology

$$H_n(f): H_n(S^n) \rightarrow H_n(S^n) \quad \text{which is}$$
$$\begin{array}{ccc} \cong & & \cong \\ \mathbb{Z} & & \mathbb{Z} \end{array}$$

determined by the number $d \in \mathbb{Z}$ to which $1 \in \mathbb{Z}$ is sent. d is defined to be the degree of f .

Useful Facts About Degrees of maps $f: S^n \rightarrow S^n$

- If f is not surjective, $\deg f = 0$. (contractibility)
- $f \simeq g \Rightarrow \deg f = \deg g$ (homotopy equivalent maps induce the same map on homology)
- $\deg f = \deg g \Rightarrow f \simeq g$ (Hopf Theorem)
- $\deg fg = \deg f \cdot \deg g$ (covariance of homology)
- From last fact, f homotopy equivalence $\Rightarrow \deg f = \pm 1$.
(e.g. homeo)
- There is a generator $[\sigma_n^N - \sigma_n^S]$ of $H_n(S^n)$ where σ_n^N / σ_n^S maps Δ^n homeomorphically (rel boundary) to the upper/lower closed hemisphere, respectively.
Picture: $\Delta \xrightarrow{\cong} \bigcirc \xrightarrow{\cong} \bigoplus_{S^{n-1}}$

consequently if $\mathcal{R}$ is the reflection $S^n \rightarrow S^n$ fixing the first n coordinates and taking the negative of the last, then $H_n(\mathcal{R})([\sigma^N - \sigma^S]) = [\mathcal{R}_* \sigma^N - \mathcal{R}_* \sigma^S] = -[\sigma^N - \sigma^S]$, hence $\deg \mathcal{R} = -1$. It follows by composition that if $\mathcal{Q}$ is the reflection through the origin, then $\deg \mathcal{Q} = (-1)^{n+1}$.

Computing Degrees — The Local Degree Formula

Let $f : S^n \rightarrow S^n$ be a map. Assume $y \in S^n$ satisfies $\bar{f}'(y) = \{x_1, \dots, x_m\}$ is finite. Assume there is a nbhd V of y and disjoint nbhds $U_1, \dots, U_m$ of $x_1, \dots, x_m$ s.t. $f(U_k) \subset V$ for all $k = 1, \dots, m$. Hence $f(U_k - x_k) \subset V - y$.

we want to examine a big commutative diagram. we have an inclusion $U_k \xrightarrow{ik} S^n$ s.t.

$ik(U_k - x_k) \subset S^n - \bar{f}'(y)$, so we have an inclusion $(U_k, U_k - x_k) \xrightarrow{ik} (S^n, S^n - \bar{f}'(y))$.

This induces a map $H_n(U_k, U_k - x_k) \xrightarrow{H_n(ik)} H_n(S^n, S^n - \bar{f}'(y))$.

Similarly we have an inclusion $(S^n, S^n - \bar{f}'(y)) \xrightarrow{jk} (S^n, S^n - x_k)$ which induces a map $H_n(S^n, S^n - \bar{f}'(y)) \xrightarrow{H_n(jk)} H_n(S^n, S^n - x_k)$. The composition of these two maps gives a map $H_n(U_k, U_k - x_k) \xrightarrow{H_n(j \circ ik)} H_n(S^n, S^n - x_k)$ which is also

just the map induced by the inclusion $j_k \circ i_k$.

claim: $H_n(j_k \circ i_k)$ is an isomorphism $H_n(U_k, U_k - x_k) \xrightarrow{\sim} H_n(S^n, S^n - x_k)$.

Proof (claim) we use Excision.

Let $X = S^n$ and $A = S^n - x_k$. Let $B = S^n - U_k$ so that

$U_k = X - B$ and $U_k - x_k = A - B$. we have

$\bar{B} = B \subset A = \text{int } A$. The Excision thm

gives the desired isomorphism. $\square$.

Let i be the inclusion $(S^n, \emptyset) \xrightarrow{i} (S^n, S^n - \bar{f}^{-1}(y))$

which induces a map $H_n(S^n) \xrightarrow{\sim} H_n(S^n, \emptyset) \xrightarrow{H_n(i)} H_n(S^n, S^n - \bar{f}^{-1}(y))$.

Again the composition $j_k \circ i$ induces a map $H_n(S^n) \xrightarrow{H_n(j_k \circ i)} H_n(S^n, S^n - x_k)$.

claim: $H_n(j_k \circ i)$ is an isomorphism $H_n(S^n) \xrightarrow{\sim} H_n(S^n, S^n - x_k)$.

Proof (claim): we use LES in relative homology.

$H_n(S^n - x_k) \xrightarrow{f} H_n(S^n) \xrightarrow{g} H_n(S^n, S^n - x_k) \xrightarrow{h} H_{n-1}(S^n - x_k) \xrightarrow{l} H_{n-1}(S^n)$. when $n > 1$, $H_n(S^n - x_k)$ and $H_{n-1}(S^n - x_k)$ are 0

since $S^n - x_k$ is contractible. when $n=1$,

$H_0(S^1 - x_k) \rightarrow H_0(S^1)$ is just the identity, hence

its kernel is 0. so $h \equiv 0$ is the zero map, hence g is an isomorphism. $\square$.

So far we have succeeded in constructing a ^{commutative} diagram: (black)

$$\begin{array}{ccccc}
 & H_n(U_k, U_k - X_k) & \xrightarrow{H_n(f)} & H_n(V, V - Y) & \\
 \swarrow \sim & \downarrow & & \downarrow ? & \\
 H_n(S^n, S^n - X_k) & \leftarrow H_n(S^n, S^n - \tilde{f}^{-1}(Y)) & \xrightarrow{H_n(f)} & H_n(S^n, S^n - Y) & \\
 \swarrow \sim & \uparrow & & \uparrow ? & \\
 & H_n(S^n) & \xrightarrow{H_n(f)} & H_n(S^n) &
 \end{array}$$

The containments $f(U_k - X_k) \subset V - Y$, $f(S^n - \tilde{f}^{-1}(Y)) \subset S^n - Y$, and the inclusions $(V, V - Y) \rightarrow (S^n - Y)$ and $(S^n, \emptyset) \rightarrow (S^n, S^n - Y)$, together w/ the same methods using Excision and LES in relative homology allow us to complete the above commutative diagram (blue).

We thus obtain identifications $H_n(U_k, U_k - X_k) \cong H_n(S^n)$ and $H_n(V, V - Y) \cong H_n(S^n)$. So the map

$$H_n(U_k, U_k - X_k) \xrightarrow{H_n(f)} H_n(V, V - Y) \text{ induced by the}$$

restriction of f to U_k is a map

$\mathbb{Z} \rightarrow \mathbb{Z}$ characterized by the image d_k of 1, which we call the local degree of f at X_k .

Thm/ (Local Degree Formula)

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Keeping w/ the current setup,

$$\deg f = \sum_k \deg f|_{X_k}$$

where $\deg f|_{X_k}$ is the local degree of f at X_k denoted previously by d_k .

Rmks/ • If each U_k is mapped

homeomorphically onto V , then

$$\deg f|_{X_k} = \pm 1 \text{ for each } k.$$

One determines the sign depending on the situation.

example/ For a detailed calculation,
see the solution to problem 3c
of the Spring 2011 geometry exam.

Poincaré Duality & Orientations (adapted from Hatcher)

Prop/ Let $x \in \mathbb{R}^n$. Then $H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \cong \mathbb{Z}$.

Proof: This is clearly true when $n=0$ and $\mathbb{R}^0 = \{x\}$.
When $n \geq 1$, we have the LES in relative homology

$$\begin{aligned} &\rightarrow H_n(\mathbb{R}^n - \{x\}) \xrightarrow{\sim H_n(S^{n-1})} H_n(\mathbb{R}^n) \rightarrow H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \xrightarrow{\sim H_n(\mathbb{R}^n, S^{n-1})} H_{n-1}(\mathbb{R}^n - \{x\}) \xrightarrow{\sim H_{n-1}(S^{n-1})} \\ &\rightarrow H_{n-1}(\mathbb{R}^n) \rightarrow H_{n-1}(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \rightarrow \end{aligned}$$

For $n \geq 1$: This becomes

$$\rightarrow 0 \rightarrow 0 \rightarrow H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \xrightarrow{\sim} \mathbb{Z} \rightarrow 0 \rightarrow H_{n-1}(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \rightarrow$$

as desired.

For $n=1$: This becomes $\begin{matrix} (x,y) \xrightarrow{\text{inclusion}} x+y \\ \text{ } \end{matrix}$

$$\rightarrow 0 \rightarrow 0 \rightarrow H_1(\mathbb{R}, \mathbb{R} - \{x\}) \rightarrow \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{f} \mathbb{Z} \rightarrow 0 \rightarrow$$

So $H_1(\mathbb{R}, \mathbb{R} - \{x\}) \cong \text{im } f = \ker g \cong \mathbb{Z}$. $\square$

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Rmk/ Let x and y be two points of $\mathbb{R}^n$, $n \geq 1$. Then there is a canonical isomorphism $H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - \{y\})$.

Indeed, let U be a ball containing x and y . Then $\mathbb{R}^n - U$ is a deformation retract of $\mathbb{R}^n - \{x\}$ and of $\mathbb{R}^n - \{y\}$.

So $H_n(\mathbb{R}^n, \mathbb{R}^n - x) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - U) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - y)$. It is canonical since the map $H_n(\mathbb{R}^n, \mathbb{R}^n - U) \rightarrow H_n(\mathbb{R}^n, \mathbb{R}^n - x)$ is just induced by inclusion.

Definition A local orientation of $\mathbb{R}^n$ at a point x is a choice of generator of $H_n(\mathbb{R}^n, \mathbb{R}^n - x) \cong \mathbb{Z}$. By the preceding remark, this forces a choice of generator of $H_n(\mathbb{R}^n, \mathbb{R}^n - y)$ for all $y \in \mathbb{R}^n$ via the canonical isomorphism $H_n(\mathbb{R}^n, \mathbb{R}^n - x) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - y)$.

Def/ An n -dimensional topological manifold M is a topological space M which is Hausdorff and satisfies that every point $x \in M$ has a neighborhood homeomorphic to $\mathbb{R}^n$.

Prop/ Let $x \in M$, M an n -dim top mfld.
Then $H_n(M, M-x) \cong \mathbb{Z}$.

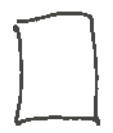
Proof: By excision:

$$H_n(M, M-x) \cong H_n(U, U-x) \cong \mathbb{Z}$$

where U is a small ball around x homeomorphic to $\mathbb{R}^n$. Here we are using the Excision thm

$$H_n(X-B, A-B) \cong H_n(X, A) \quad (\bar{B} \subset A)$$

w/ $X = M, A = M-x, B = U^c$.



Def/ A local orientation of an n -dim
top mfld M at a point x in M is the
choice of a generator of $H_n(M, M-x)$.

Props/ • For the same reason as for $\mathbb{R}^m$,
a local orientation of M at x forces
the choice of a generator of $H_n(M, M-y)$
at each y in a small ball around x .

- There are only two orientations at x .
- Following the next def, a connected mfld has only two orientations.

Def/ A global orientation of an n -dim
top mfld M is the choice of a local
orientation at each $x \in M$ s.t. these
choices coincide for all y w/ the local
orientation at y forced by the local orientation at any nearby x .

M is orientable if it admits a global orientation.

Prop/ Let M be a connected
 n -dim top mfld. If $\pi_1(M)$
does not have a subgroup of index
2, then M is orientable.

In particular, M is orientable if
it is simply connected.

Proof/ Omitted. $\square$.

Thm/ Let M be a compact connected
 n -dim top mfld.

- (a) If M is orientable, the projection-induced map
 $H_n(M) \rightarrow H_n(M, M-x) \cong \mathbb{Z}$ is an isomorphism for all $x \in M$.
- (b) If M is not orientable, the same map
 $H_n(M) \rightarrow H_n(M, M-x) \cong \mathbb{Z}$ is inj and is the zero map;
that is, $H_n(M) = 0$.
- (c) $H_k(M) = 0$ for $k > \dim M$.

Proof: Omitted. $\square$.

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Thm/

Let M be a smooth manifold.

Then M is orientable as a manifold
iff M is orientable as a top manifold.

Proof/ Omitted.

Thm/ (Poincaré Duality)

Recall the def of the k^{th} cohomology
group $H^k(X; \mathbb{Z})$ of a topological space.

Let $X \equiv M$ be an n -dim compact top mfld.

For each $k = 0, 1, \dots, n$, $H^k(M; \mathbb{Z}) = H_{n-k}(M; \mathbb{Z})$.

Proof / Omitted.

Reduced Homology

Def / Let X be a topological space.

The augmented chain complex associated to X (with integer coefficients) is the chain complex

$$\cdots \rightarrow C_k(X) \xrightarrow{\partial_k \equiv \tilde{\partial}_k} C_{k-1}(X) \xrightarrow{\partial_{k-1} \equiv \tilde{\partial}_{k-1}} \cdots \rightarrow C_1(X) \xrightarrow{\partial_1 \equiv \tilde{\partial}_1} C_0(X) \xrightarrow{\varepsilon \equiv \tilde{\partial}_0} \mathbb{Z} \rightarrow 0 \rightarrow \cdots$$

where ∂_k is the usual boundary map, and

$\varepsilon: C_0(X) \rightarrow \mathbb{Z}$ is defined by

$$\varepsilon\left(\sum_{i=1}^n a_i x_i\right) = \sum_{i=1}^n a_i.$$

Then it is easy to show $\varepsilon \circ \partial_1 = 0$, and so we define the k^{th} relative homology group of X to

$$\text{be } \tilde{H}_k(X; \mathbb{Z}) = \ker \tilde{\partial}_k / \text{im } \tilde{\partial}_{k+1}.$$

Thm/ For $k > 0$, $H_k(X, \mathbb{Z}) \cong \tilde{H}_k(X, \mathbb{Z})$. 13

For $k=0$, $H_0(X, \mathbb{Z}) \cong \mathbb{Z} \oplus \tilde{H}_k(X, \mathbb{Z})$.

Proof/ Omitted. $\square$.

example/ For a detailed example where you have to play around w/ long exact sequences and reduced homology, see problem 4 of the Spring 2011 Geometry EXAM.

Seifert - van Kampen Theorems (adapted from Bonahon)

Def/ Let A and B be groups. The free product $A * B$ of A and B is the group of all

finite sequences $X_1 X_2 \cdots X_n$ s.t. (i) $X_i \in A$ or B , $X_i \neq 1_A$ or 1_B ; and (ii) if $X_i \in A$ (resp. B) then $X_{i+1} \in B$ (resp. A). The group law

is given by concatenation and simplification.
The identity is given by the empty sequence.

Def/ The free group F_n on n generators is
the group $\underbrace{\mathbb{Z} * \mathbb{Z} * \dots * \mathbb{Z}}_{n \text{ copies}}$ where $\mathbb{Z}$ is
viewed multiplicatively, i.e. $\mathbb{Z} = \langle x \rangle$.

Def/ (Group Presentations)

Let F_n be the free group generated by $x_1, \dots, x_n$.

Let $r_1, r_2, \dots, r_m \in F_n$.

The group presented by the generators x_i and

relations r_i , denoted $\langle x_1, \dots, x_n \mid r_1 = \dots = r_m = 1 \rangle$,

is defined as the quotient F_n / K where

$K = \langle g r_i g^{-1} : i = 1, \dots, m, g \in F_n \rangle$ is the smallest

normal subgroup of F_n containing the r_i . (K is
also called the subgroup 'normally generated' by the r_i .)

Def/ A finite presentation for a group G 14
is an isomorphism $G \cong F_n / K$.

(Culture)
Thm/ (A.A. Markov)

There exists no algorithm which given $\varepsilon_1, \dots, \varepsilon_m \in F_n$ determines whether the group $G = \langle X_1, \dots, X_n \mid \varepsilon_1 = \dots = \varepsilon_m = 1 \rangle$ is the trivial group or not.

Def/ (A amalgamated Products)

Let A, B, C be groups.

Let $i_A: C \rightarrow A, i_B: C \rightarrow B$ be group homomorphisms.

The amalgamated product $A *_C B$ of A and B amalgamated along C via the group homomorphisms i_A and i_B is

the quotient $A *_C B / K$ where K is the subgroup normally generated by $\{ i_A(c) i_B(c)^{-1} \in A *_C B : c \in C \}$.

Rmks/ • The amalgamated product along i_A, i_B is the pushout in the category of groups. Specifically, first note that there are natural maps j_A, j_B defined by

$$\begin{array}{ccc} A & \xrightarrow{j_A} & A * B \\ B & \xrightarrow{j_B} & A * B \end{array} \rightarrow A * B / K$$

Then for any group D and s.t. $C \rightarrow A \rightarrow D$ is a commutative diagram

any group homomorphisms $A \rightarrow D, B \rightarrow D \vee$ there is a unique group homomorphism $A *_C B \xrightarrow{\varphi} D$ making the following diagram commute:

$$\begin{array}{ccccc} & & A & & \\ & \nearrow i_A & & \searrow j_A & \\ C & & & & \\ & \searrow i_B & & \nearrow j_B & \\ & & B & & \\ & & & & A *_C B \xrightarrow{\exists! \varphi} D \end{array}$$

- For all $c \in C$, $\overline{i_A(c)} = \overline{i_B(c)} \in A *_C B$.
- $\star$ In practice, if $A = \langle a_1, \dots, a_m \mid r_1 = \dots = r_k = 1 \rangle$ and $B = \langle b_1, \dots, b_n \mid s_1 = \dots = s_\ell = 1 \rangle$ and $C = \langle c_1, \dots, c_p \rangle$, then $A * B = \langle a_1, \dots, a_m, b_1, \dots, b_n \mid r_1 = \dots = r_k = s_1 = \dots = s_\ell = 1 \rangle$ and $A *_C B = \langle a_1, \dots, a_m, b_1, \dots, b_n \mid r_1 = \dots = r_k = s_1 = \dots = s_\ell = i_A(c_1) i_B(c_1)^{-1} = \dots = i_A(c_p) i_B(c_p)^{-1} = 1 \rangle$.

Rmks/ • Always ask yourself what $i_A: C \rightarrow A$

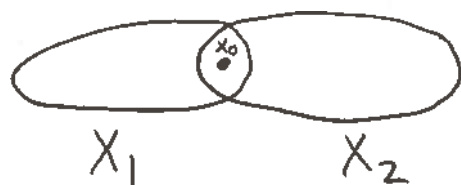
and $i_B: C \rightarrow B$ are. Without this information,

notations like $\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}$ are meaningless.

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Thm/ (Seifert - van Kampen I)

Setup:



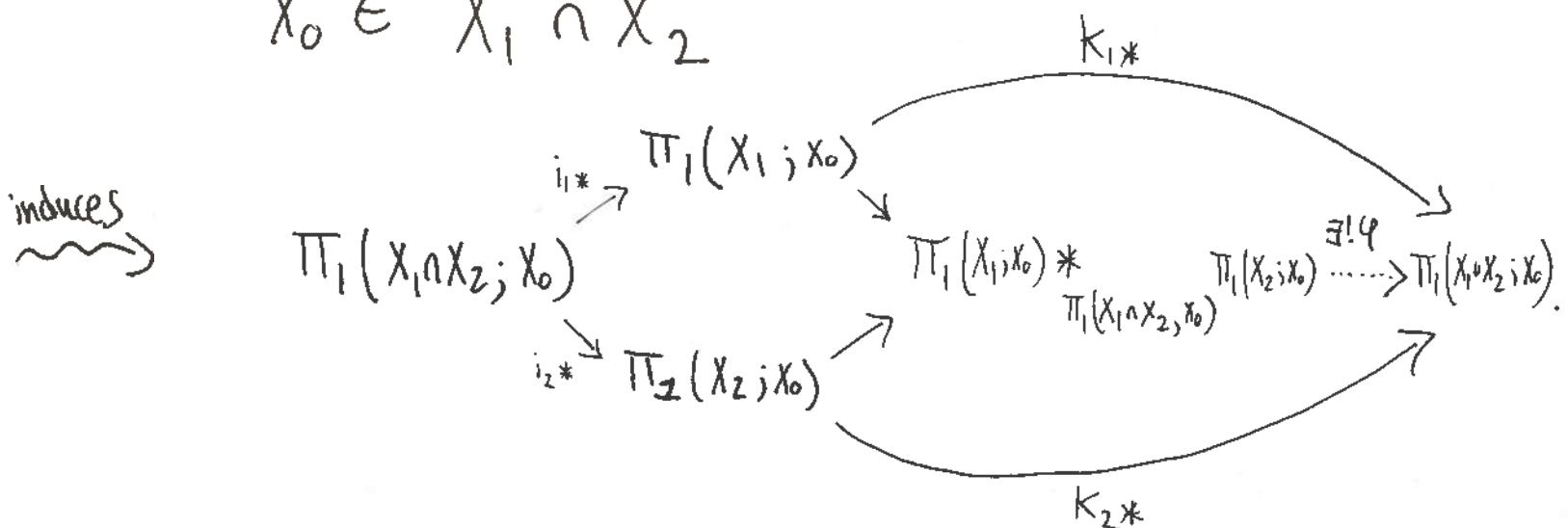
$$i_1: X_1 \cap X_2 \rightarrow X_1$$

$$k_1: X_1 \rightarrow X_1 \cup X_2$$

$$i_2: X_1 \cap X_2 \rightarrow X_2$$

$$k_2: X_2 \rightarrow X_1 \cup X_2$$

$$x_0 \in X_1 \cap X_2$$



Here we are using that $k_{1*} \circ i_{1*} = k_{2*} \circ i_{2*} \Rightarrow k_{1*} \circ i_{1*} = k_{2*} \circ i_{2*}$, so that the outer diagram commutes and the UMP of the pushout can be used.

The result: If X_1, X_2 are open in $X_1 \cup X_2$, $x_0 \in X_1 \cap X_2$, and $X_1 \cap X_2$ is path connected, then the induced homomorphism ψ gives a natural isomorphism $\pi_1(X_1; x_0) *_{\pi_1(X_1 \cap X_2; x_0)} \pi_1(X_2; x_0) \cong \pi_1(X_1 \cup X_2; x_0)$.

Thm / (Seifert - van Kampen II)

setup: The same as for Svk I.

result: If X_1, X_2 are closed in

$X_1 \cup X_2$, $x_0 \in X_1 \cap X_2$, X_1 and X_2 and $X_1 \cap X_2$ are path connected, and $X_1 \cap X_2$ is a neighborhood deformation retract in both X_1 and X_2 , then

the same conclusion as in Svk is true.

Covering Spaces (adapted from Banahan)

Thm/ (path lifting property)

Let $p: \tilde{X} \rightarrow X$ be a covering space.

Let $\alpha: [0,1] \rightarrow X$ be a path.

Let $X_0 = \alpha(0)$ and let $\tilde{X}_0 \in p^{-1}(X_0)$.

Then there is a unique path $\tilde{\alpha}: [0,1] \rightarrow \tilde{X}$ s.t.
 $\tilde{\alpha}(0) = \tilde{X}_0$ and $\alpha = p \circ \tilde{\alpha}$.

Thm/ (homotopy lifting property)

Let $p: \tilde{X} \rightarrow X$ be a covering space.

Let α, β be two paths in X s.t. $\alpha(0) = \beta(0) = X_0$.

Let $\tilde{X}_0 \in p^{-1}(X_0)$. Let $\tilde{\alpha}, \tilde{\beta}$ the unique lifts of α, β w/ $\tilde{\alpha}(0) = \tilde{\beta}(0) = \tilde{X}_0$.

If $\alpha \sim \beta$ by a path homotopy $H: I \times I \rightarrow X$, then

H lifts to a unique path homotopy $\tilde{H}: I \times I \rightarrow \tilde{X}$ between $\tilde{\alpha}$ and $\tilde{\beta}$ satisfying $p \circ \tilde{H} = H$.

Cor/ $\tilde{\alpha}$ and $\tilde{\beta}$ end at the same point in $\tilde{X}$.

Thm/ (lifting criterion)

Let $p: \tilde{X} \rightarrow X$ be a covering space.

Let $f: Y \rightarrow X$. Let Y be path connected and locally path connected.

Let $y_0 \in Y$, $x_0 = f(y_0) \in X$, $\tilde{x}_0 \in p^{-1}(x_0) \subset \tilde{X}$.

Then there exists a unique lift $\tilde{f}: Y \rightarrow \tilde{X}$ satisfying $p \circ \tilde{f} = f$
and $\tilde{f}(y_0) = \tilde{x}_0$ if and only if $f_*(\pi_1(Y; y_0)) \subset p_*(\pi_1(\tilde{X}; \tilde{x}_0)) \subset \pi_1(X; x_0)$.

Def/ A space X is simply connected if X is
p.c., locally p.c., and $\pi_1(X; x_0) = 1$.

Cor/ If X is simply connected, then the lifting
criterion is always met.

Def/ The covering spaces $p: \tilde{X} \rightarrow X$ and $p': \tilde{X}' \rightarrow X$
are isomorphic if there exists a homeomorphism $\varphi: \tilde{X} \rightarrow \tilde{X}'$
s.t. $\tilde{X} \xrightarrow{\varphi} \tilde{X}'$ commutes, i.e. $p' \circ \varphi = p$.

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\varphi} & \tilde{X}' \\ p \searrow & & \swarrow p' \\ & X & \end{array}$$

Thm/ Let $p: \tilde{X} \rightarrow X$ and $p': \tilde{X}' \rightarrow X$ be

covering spaces w/ X p.c. and locally p.c. $\tilde{X}, \tilde{X}'$ p.c.

Let $x_0 \in X$, $\tilde{x}_0 \in p^{-1}(x_0) \subset \tilde{X}$, and $\tilde{x}'_0 \in (p')^{-1}(x_0) \subset \tilde{X}'$.

Then there exists an isomorphism $\varphi: \tilde{X} \rightarrow \tilde{X}'$ sending $\tilde{x}_0 \mapsto \tilde{x}'_0$ if and only if $p_*(\pi_1(\tilde{X}; \tilde{x}_0)) = p'_*(\pi_1(\tilde{X}'; \tilde{x}'_0)) \subset \pi_1(X; x_0)$.

Q: What if we remove the condition that $\varphi(\tilde{x}_0) = \tilde{x}'_0$?

A: Thm/ Let $p: \tilde{X} \rightarrow X$ and $p': \tilde{X}' \rightarrow X$ be covering spaces w/ X p.c. and locally p.c. $\tilde{X}, \tilde{X}'$ p.c.

Let $x_0 \in X$, $\tilde{x}_0 \in p^{-1}(x_0) \subset \tilde{X}$, and $\tilde{x}'_0 \in (p')^{-1}(x_0) \subset \tilde{X}'$.

Then there exists an isomorphism $\varphi: \tilde{X} \rightarrow \tilde{X}'$ if and only if $p'_*(\pi_1(\tilde{X}'; \tilde{x}'_0))$ is a conjugate of $p_*(\pi_1(\tilde{X}; \tilde{x}_0))$ in $\pi_1(X; x_0)$.

Rmk/ • This theorem says there is an injection of the collection $\{\text{isomorphism classes of } \text{path-connected covering spaces rel. base point}\} \xrightarrow{p_*} \{\text{subgroups of } \pi_1(X; x_0)\}$ when X is p.c. and locally p.c.

• Q: when is this a 1-1 correspondence?

A: Def/ X is locally simply connected if
for every $x \in X$ and for each nbhd $U \ni x$ there exists
a nbhd $V \ni x$ s.t. $V \subset U$, V p.c., locally p.c., $\pi_1(V; x) = 1$.

Def/ X is semi-locally simply connected if
for every $x \in X$ and for each nbhd $U \ni x$ there exists a
nbhd $V \ni x$ s.t. $V \subset U$, V p.c., locally p.c., and the homomorphism
 $i_* : \pi_1(V; x) \rightarrow \pi_1(X; x)$ induced by the inclusion $V \xrightarrow{i} X$ is
the trivial homomorphism.

Remarks/ • So locally simply connected $\Rightarrow$ semi-locally simply connected
• So all topological manifolds are locally simply connected $\Rightarrow$ semi-locally simply connected.

Thm/ When X is path connected, locally path connected,
and semi-locally simply connected, then the
correspondence between 'path connected covering spaces' and subgroups of
 $\pi_1(X; x_0)$ is a 1-1 correspondence.

Cor/ (of the proof)

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If X is p.c., locally p.c., and semi-locally simply connected, then there is a unique isomorphism class of path connected coverings $\tilde{X}$ (rel. base point or not) s.t. $\pi_1(\tilde{X}, \tilde{x}_0) = 1$.

Def/ Such an $\tilde{X}$ is called the universal cover of X . (Note: This whole time, because of the def of covering space, $\tilde{X}$ p.c. $\iff \tilde{X}$ p.c. & locally p.c. With this in mind, it is convenient to observe the universal cover $\tilde{X}$ is simply connected.)

Rmks/ • We put a partial ordering on p.c. coverings so that this 1-1 correspondence becomes inclusion reversing. Specifically, we say $(\tilde{X}, p) \geq (\tilde{X}', p')$ if p lifts w.r.t. p' , i.e. there is $q: \tilde{X} \rightarrow \tilde{X}'$ s.t. $p' \circ q = p$ so that the following diagram commutes

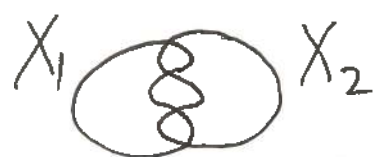
$$\begin{array}{ccc} \tilde{X} & \xrightarrow{q} & \tilde{X}' \\ p \downarrow & & \downarrow p' \\ X & & X \end{array}$$

By the Lifting Criterion, $\tilde{X} \geq \tilde{X}'$ iff $p_*(\pi_1(\tilde{X}; \tilde{x}_0)) \leq p'_*(\pi_1(\tilde{X}'; \tilde{x}'_0))$. In particular, the universal cover $\tilde{X}$ is the 'largest' cover.

Mayer - Vietoris Theorems

Thm / (Mayer - Vietoris I)

Let X_1, X_2 open $\subset X_1 \cup X_2$.



$$i_1: X_1 \cap X_2 \rightarrow X_1 \quad j_1: X_1 \rightarrow X_1 \cup X_2$$

$$i_2: X_1 \cap X_2 \rightarrow X_2 \quad j_2: X_2 \rightarrow X_1 \cup X_2$$

Then there is a LES

$$\rightarrow H_{k+1}(X_1 \cup X_2) \xrightarrow{\delta_{k+1}} H_k(X_1 \cap X_2) \xrightarrow{(-H_k(i_1)) \oplus H_k(i_2)} H_k(X_1) \oplus H_k(X_2) \xrightarrow{H_k(j_1) + H_k(j_2)} H_k(X_1 \cup X_2) \xrightarrow{\delta_k} H_{k-1}(X_1 \cap X_2) \rightarrow$$

Thm / (Mayer - Vietoris II)

Let X_1, X_2 closed $\subset X_1 \cup X_2$

X_1, X_2 a nbhd deformation retract in $X_1 \cup X_2$.

Then get same LES as in MV I.

Relative Homology

& Excision



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when $n=1$

$$H(i_+) \oplus (-H(i_-))$$

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$$H_0(S^0) \rightarrow H_0(B_+^1) \oplus H_0(B_-^1)$$

$$R \oplus R \xrightarrow{f} R \oplus R$$

$$(x, y) \mapsto (x+y, -x-y)$$

$$i_+ : S^0 \hookrightarrow B_+^1$$

$$i_- : S^0 \hookrightarrow B_-^1$$

arbitrary elt of $H_0(S^0)$ is
class of elt in one component +
class of elt in another component.

$$[x + y] = [x] + [y]$$

$$\mapsto [x+y] - [x] - [y]$$

$$\ker f = \{ (x, y) : x+y=0 \}$$

$$= \{ (x, -x) : x \in R \} \cong R.$$

$$\begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline S^0 & R \oplus R & 0 & 0 & 0 & 0 \\ S^1 & R & R & 0 & 0 & 0 \\ S^2 & R & 0 & R & 0 & 0 \end{array}$$

$$\alpha \rightarrow H_1(S^1) \rightarrow H_0(S^0) \rightarrow H_0(B_+^1) \oplus H_0(B_-^1)$$

$$H_1(S^1) \cong \ker f \cong R$$

$$H_1(S^1) \cong R$$

Conclusion

~~Not valid~~

$$H_1(S^n) \cong \begin{cases} 0 & n > 1 \\ R & n = 1 \end{cases}$$

$$H_k(S^n) \cong H_k(S^{n-1}) \cong \dots \cong H_1(S^{n-k+1})$$

If $k > n$, $H_k(S^n) = H_k(S^0) = 0$.

If $k < n$, $n-k+1 > 1 \Rightarrow H_k(S^n) = 0$. If $k > n$, $H_k(S^n) \cong H_{k-n}(S^0) = 0$.

Relative Homology

Setup: X space, $A \subset X$ subspace. $i_A: A \rightarrow X$ inclusion.

$$S_n(X) = \{ \text{simplices } \sigma_n: \Delta_n \rightarrow X \}$$

we may view $S_n(A) \subset S_n(X)$ by $\sigma_n: \Delta_n \rightarrow A \rightarrow X$.

$$C_n(X) = \left\{ \text{chains } \sum_{i=1}^m a_i \sigma_i : a_i \in \mathbb{R}, \sigma_i \in S_n(X) \right\}$$

so we may view $C_n(A) \subset C_n(X)$. Moreover, $C_n(A)$ is a submodule.

Consider the quotient $C_n(X, A) := C_n(X)/C_n(A)$.

The boundary map $\partial_n: C_n(X) \rightarrow C_{n-1}(X)$ defined by

$$\partial_n(\sigma) = \sum_{i=0}^n (-1)^i \sigma \circ f_i \quad \text{for } \sigma \in S_n(X) \text{ and extended linearly,}$$

where $f_i: \Delta_{n-1} \rightarrow \Delta_n$ is the i^{th} face map, restricts to a

boundary map $C_n(A) \rightarrow C_{n-1}(A)$. Hence we get a map $\bar{\partial}_n: C_n(X, A) \rightarrow C_{n-1}(X, A)$.

$$\begin{array}{ccccc} C_n(X) & \rightarrow & C_{n-1}(X) & \rightarrow & C_{n-1}(X)/C_{n-1}(A) \\ & \searrow & & & \nearrow \\ & C_n(X)/C_n(A) & \cdots & & \end{array}$$

And $\bar{\partial}_{n-1} \circ \bar{\partial}_n = 0$.

$$\begin{array}{ccccc} & & 0 & & \\ & \searrow & & \searrow & \\ C_n(X) & \rightarrow & C_{n-1}(X) & \rightarrow & C_{n-2}(X) \\ \downarrow & & \downarrow & & \downarrow \\ C_n(X, A) & \rightarrow & C_{n-1}(X, A) & \rightarrow & C_{n-2}(X, A) \\ & \searrow & & \searrow & \\ & 0 & & & \end{array}$$

Define the n^{th} relative homology $\mathbb{R}$ -module of X w.r.t. A as $H_n(X, A) := \ker \bar{\partial}_n / \text{im } \bar{\partial}_{n-1}$.

e.g. $A = \emptyset$. $C_n(\emptyset) = 0$. $C_n(X, \emptyset) = C_n(X)$. $H_n(X, \emptyset) = H_n(X)$.

Rmk/ we want a more geometric picture.

Let $Z_n(X, A) := \{ C_n \in C_n(X) : \partial_n C_n \in C_{n-1}(A) \}$

and let $B_n(X, A) := \left\{ C_n \in C_n(X) : \begin{array}{l} \exists C_{n+1} \in C_{n+1}(X) \text{ and} \\ \exists a_n \in C_n(A) \end{array} \right\}$.

Clearly $B_n(X, A) \subset Z_n(X, A)$: $\partial_n(a_n + \partial_{n+1} C_{n+1}) = \partial_n(a_n) \in C_{n-1}(A)$. $\checkmark$ s.t. $C_n = a_n + \partial_{n+1} C_{n+1}$.

Proposition: $H_n(X, A) \cong Z_n(X, A) / B_n(X, A)$.

Proof/ Set $\bar{Z}_n(X, A) := \ker \bar{\partial}_n \subset C_n(X) / C_n(A)$.

Set $\bar{B}_n(X, A) := \text{im } \bar{\partial}_{n+1} \subset C_{n+1}(X) / C_{n+1}(A)$.

Let π be the natural projection $C_n(X) \rightarrow C_n(X) / C_n(A)$.

Since $\bar{Z}_n(X, A) = \ker \bar{\partial}_n$, $\pi(Z_n(X, A)) \subset \bar{Z}_n(X, A)$:

Indeed, if $C_n \in C_n(X)$ is in $Z_n(X, A)$, then

$$\bar{\partial}_n(\pi(C_n)) = \pi(\overbrace{\partial_n(C_n)}^{\in C_{n-1}(A)}) = 0 \in C_{n-1}(X) / C_{n-1}(A).$$

And $\pi(B_n(X, A)) \subset \bar{B}_n(X, A)$:

Indeed, if $C_n \in C_n(X)$ is in $B_n(X, A)$, w/ $C_n = \partial_{n+1} \overbrace{C_{n+1}}^{C_{n+1}(X)} + \overbrace{a_n}^{C_n(A)}$,

Then $\bar{\partial}_{n+1}(\pi(C_{n+1})) = \pi(\partial_{n+1}(C_{n+1})) = \pi(C_n - a_n) = \pi(C_n) \in C_n(X) / C_n(A)$.

There we have an ordered map $\bar{\pi} : \frac{Z_n(X, A)}{B_n(X, A)} \rightarrow \frac{\bar{Z}_n(X, A)}{\bar{B}_n(X, A)}$

$$\begin{aligned} Z_n(X, A) &\xrightarrow{\pi} \bar{Z}_n(X, A) \rightarrow \bar{Z}_n(X, A) / \bar{B}_n(X, A) \\ &\searrow Z_n(X, A) / B_n(X, A) \xrightarrow{\bar{\pi}} \end{aligned}$$

claim: $\bar{\pi}^{-1} \bar{B}_n(X, A) = B_n(X, A)$, i.e. $\bar{\pi}$ is injective.

Let $C \in \bar{\pi}^{-1} \bar{B}_n(X, A) \subset C_n(X)$.

so $\pi(C) \in \bar{B}_n(X, A)$. so $\exists C_{n+1} \in C_{n+1}(X)$ s.t.

$$\pi(C) = \pi(\partial_{n+1}(C_{n+1})) \in C_n(X) / C_n(A).$$

so $C - \partial_{n+1}(C_{n+1}) = a_n \in C_n(A)$. Gives $\subseteq$.

Conversely, if $C = \partial_{n+1}(C_{n+1}) + a_n$,

$$\text{then } \pi(C) = \pi(\partial_{n+1}(C_{n+1})) = \bar{\partial}_{n+1} \pi(C_{n+1}).$$

so $\pi(C) \in \bar{B}_n(X, A)$. Gives $\supseteq$. ✓

claim: $\bar{\pi}$ is surjective. $C_n(X) / C_n(A)$

~~we know $\pi(Z_n(X, A)) \subset \bar{Z}_n(X, A)$.~~

we know $\pi(Z_n(X, A)) \subset \bar{Z}_n(X, A)$. ~~we know $\pi(Z_n(X, A)) \subset \bar{Z}_n(X, A)$.~~

~~we know $\pi(Z_n(X, A)) \subset \bar{Z}_n(X, A)$.~~ is the converse true?

Let $\bar{\partial}_n(\pi(C)) = 0 \in C_{n-1}(X) / C_{n-1}(A)$. So $\pi(\partial_n(C)) = 0$.

So $\partial_n(C) \in C_{n-1}(A)$. So $C \in Z_n(X, A)$. ✓ so have $\supseteq$.

This finishes the proof of the Proposition. $\square$.

Thm/ X p.c. Then $H_0(X, A) = \begin{cases} 0 & \text{if } A \neq \emptyset \\ \mathbb{R} & \text{if } A = \emptyset \end{cases}$ 3

Proof/ we know the result is true when $A = \emptyset$.

Let $A \neq \emptyset$. ~~we need to show~~ we need to show $Z_0(X, A) \subset B_0(X, A)$.

In fact, we show $B_0(X, A) = C_0(X)$.

Fix $x_0 \in A$. Let $c \in C_0(X)$. $c = \sum_{i=1}^k a_i x_i$.

For each i , choose $\sigma_i : \Delta_1 \rightarrow X$ s.t. $\sigma_i(1) = x_i$ and $\sigma_i(0) = x_0$.

(since X p.c.)

Then $\partial \sigma_i = x_i - x_0$.

$$\begin{aligned} \text{so } c &= \underbrace{\sum_{i=1}^k a_i \partial \sigma_i}_{\in \partial C_1(X)} - \underbrace{x_0 \sum_{i=1}^k a_i}_{\in C_0(A)} \quad \text{so } c \in B_0(X, A). \\ &= \partial \left(\sum_{i=1}^k a_i \sigma_i \right) \in C_0(A) \end{aligned} \quad \square$$

Thm/ If $\{X_i\}$ are the components of X ,
then $H_n(X, A) = \bigoplus_i H_n(X_i, A_i)$ w/ $A_i \equiv X_i \cap A$.

Rmk/ we have a SES of chain complexes

$$\begin{array}{ccccccc} 0 & \rightarrow & C_n^{\downarrow}(A) & \xrightarrow{\text{in}} & C_n^{\downarrow}(X) & \xrightarrow{\pi_n} & C_n^{\downarrow}(X, A) \rightarrow 0 \\ & & \downarrow \partial_n & & \downarrow \partial_n & & \downarrow \partial_n \\ 0 & \rightarrow & C_{n-1}(A) & \rightarrow & C_{n-1}(X) & \rightarrow & C_{n-1}(X, A) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \end{array}$$

which we may write $0 \rightarrow C(A) \xrightarrow{i} C(X) \xrightarrow{\pi} C(X, A) \rightarrow 0$.

Therefore, we get a LES in homology

$$\begin{array}{c} \rightarrow H_{n+1}(X, A) \\ \curvearrowright H_n(A) \xrightarrow{H_n(i)} H_n(X) \xrightarrow{H_n(\pi)} H_n(X, A) \curvearrowright \\ \curvearrowleft H_{n-1}(A) \rightarrow \end{array} \quad S_n$$

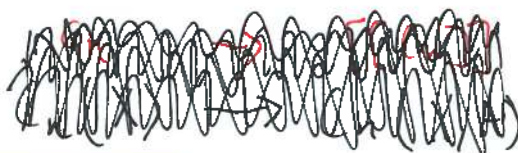
we can understand S_n concretely:

Let $[C_n] \in H_n(X, A) = Z_n(X, A) / B_n(X, A)$.

so $C_n \in C_n(X)$, $\partial C_n \in C_{n-1}(A)$.

$[[C_n]] \in C_n(X, A) = C_n(X) / C_n(A)$.

where in $C_{n-1}(A)$ is the natural place for C_n to go? $\partial C_n \in \ker \partial_{n-1}$ of course.



$$S_n([C_n]) = [\partial C_n] \in H_{n-1}(A)$$

Why is $[\partial C_n] \neq 0$ in $H_{n-1}(A)$?

Why is $[\partial C_n] = 0$ in $H_{n-1}(X)$?

Because $C_n \in C_n(X)$.

But C_n need not be in $C_n(A)$.

4

we get the LES in relative homology

If $p > 1$: Then $H_p(B^{n+1}) \rightarrow H_p(B^{n+1}, S^n) \rightarrow H_{p-1}(S^n) \rightarrow H_{p-1}(B^{n+1})$

If $p=1$; if $n > 0$:

$$H_1(B^{n+1}, S^n) \cong \text{im } \delta_1 \cong \ker H_0(i) : \mathbb{R} \rightarrow \mathbb{R}$$

$$H_1(B^{n+1}, S^n) \cong 0$$

$$(n > 0, p = 1)$$

So the map $R \rightarrow R$ is the identity map, in particular, injective

If $p=1$; if $n=0$:

$$\begin{array}{ccccccc}
 H_1(B^{n+1}) & \rightarrow & H_1(B^{n+1}, S^n) & \xrightarrow{\delta_1} & H_0(S^n) & \xrightarrow{H_0(i)} & H_0(B^{n+1}) \rightarrow H_0(B^{n+1}, S^n) \\
 \bigcirc & & & & R \oplus R & & R \quad \bigcirc
 \end{array}$$

$\begin{array}{cc} p.c. & \neq \emptyset \\ \downarrow & \downarrow \end{array}$

$$H_1(B^{n+1}, S^n) \cong \text{im } \delta_1 \cong \ker H_0(i) : R \oplus R \rightarrow R$$

$(x, y) \mapsto x + y$

$$\ker H_0(i) = \{ (x, -x) \} \subset R \oplus R \cong R.$$

$$H_1(B^1, S^0) \cong R \quad (n=0, p=1)$$

Next:

$$B_-^n = \{ x \in S^n : x_{n+1} \leq 0 \}$$



$\cong B^n$ by projection along x_{n+1} axis.

Let $X = S^n$ and let $A = B_-^n$.

~~Let $p > 1$:~~

Get LES in relative homology.

$$\rightarrow H_{p+1}(S^n, B_-^n)$$

$$\rightarrow H_p(B_-^n) \rightarrow H_p(S^n) \rightarrow H_p(S^n, B_-^n)$$

$$\rightarrow H_{p-1}(B_-^n) \rightarrow \delta_p$$

If $p > 1$:

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$$H_p(B_-^n) \rightarrow H_p(S^n) \rightarrow H_p(S^n, B_-^n) \rightarrow H_{p-1}(B_-^n)$$

$$H_p(S^n) \cong H_p(S^n, B_-^n) \quad (n \geq 0, p > 1)$$

If $p=1, n \geq 0$:

$$H_1(B_-^n) \rightarrow H_1(S^n) \rightarrow H_1(S^n, B_-^n) \xrightarrow{\delta_1} H_0(B_-^n) \xrightarrow{H_0(i)} H_0(S^n)$$



R or $R \oplus R$

either $R \rightarrow R$
 $x \mapsto x$

or $R \rightarrow R \oplus R$
 $x \mapsto (x, 0)$

$\therefore H_0(i)$
injective

$$H_1(S^n) \cong \ker \delta_1, \quad \text{im } \delta_1 \cong \ker H_0(i) = 0 \Rightarrow \delta_1 = 0.$$

$$\Rightarrow H_1(S^n) \cong H_1(S^n, B_-^n) \quad (p=1, n \geq 0)$$



Def/ Let $A \subset X$ and $B \subset Y$.

6

By a map $f: (X, A) \rightarrow (Y, B)$

it is meant a continuous map $f: X \rightarrow Y$ s.t.

$$f(A) \subset B.$$

Rmk/ Let $f: (X, A) \rightarrow (Y, B)$.

$$f: X \rightarrow Y \rightsquigarrow C_n(f): C_n(X) \rightarrow C_n(Y)$$

defined by extending $\sigma: \Delta_n \rightarrow X \rightsquigarrow \tilde{\sigma}: \Delta_n \rightarrow X \xrightarrow{f} Y$.

Since $f(A) \subset B$, $C_n(f)(C_n(A)) \subset C_n(B)$.

So we get an induced map

$$\overline{C_n(f)}: C_n(X, A) \rightarrow C_n(Y, B)$$

which commutes w/ boundaries, inducing a chain map and thus a map at the level of homology:

$$H_n(f): H_n(X, A) \rightarrow H_n(Y, B).$$

Thm / (Excision Theorem)

Def/ Let $B \subset A \subset X$. Then the map
 $(X-B, A-B) \xrightarrow{i} (X, A)$

induces a map on homology

$$H_n(i) : H_n(X-B, A-B) \rightarrow H_n(X, A).$$

We say the inclusion i is an excision,

or that B can be excised from (X, A) ,

if $H_n(i)$ gives an iso $H_n(X-B, A-B) \cong H_n(X, A)$ for all n .

[The theorem is that if $\bar{B} \subset \text{int } A$
(i.e. " B sits deeply inside A "),
then B can be excised from (X, A) .]

~~Key to~~

Key Lemma :

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Let $\mathcal{U}$ be an open covering of X .

Let $S_n^{\mathcal{U}}(X) = \left\{ \text{simplices } \sigma: \Delta_n \rightarrow X \text{ such that } \sigma \text{ maps } \Delta_n \text{ into one of the } U \in \mathcal{U} \right\}$.

Let $C_n^{\mathcal{U}}(X) = \text{free } R\text{-module generated by } S_n^{\mathcal{U}}(X)$.

The boundary ∂ maps $C_n^{\mathcal{U}}(X) \rightarrow C_{n-1}^{\mathcal{U}}(X)$

so we get homology groups $H_n^{\mathcal{U}}(X)$,

as well as an inclusion $H_n^{\mathcal{U}}(X) \hookrightarrow H_n(X)$.

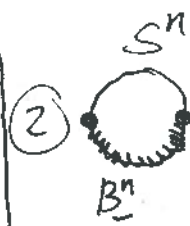
[The theorem is that this inclusion gives an iso $H_n^{\mathcal{U}}(X) \cong H_n(X)$]

e.g. Return to the computation of $H_k(S^n)$.

Summary



$$\begin{aligned} H_p(B^{n+1}, S^n) &\cong H_{p-1}(S^n) \\ (p \geq 1, n \geq 0) \\ H_1(B^{n+1}, S^n) &\cong \begin{cases} 0, & n > 0 \\ \mathbb{R}, & n = 0 \end{cases} \end{aligned}$$



$$\begin{aligned} H_p(S^n) &\cong H_p(S^n, B^n) \\ (p \geq 1, n \geq 0) \end{aligned}$$

Lemma: $\text{int}(B_-^n)$ can be excised from (S^n, B_-^n) ,

i.e. the map on homology $H_k(S^n - \text{int}(B_-^n), B_-^n - \text{int}(B_-^n)) \xrightarrow{H_k(B_+^n, S^{n-1})} H_k(S^n, B_-^n)$
induced by the inclusion $(S^n - \text{int}(B_-^n), B_-^n - \text{int}(B_-^n)) \rightarrow (S^n, B_-^n)$.
Say for $n \geq 1$.

Note: we can't immediately use the Excision Theorem
because $B_-^n = \overline{\text{int}(B_-^n)} \not\subset \text{int}(B_-^n)$.

Proof: Let $B \subset \text{int}(B_-^n)$ be defined by $B = \{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} :$

$x_0^2 + x_1^2 + \dots + x_n^2 = 1$ and $x_n \leq -\frac{1}{2}\}$. Then $\bar{B} \subset \text{int}(B_-^n)$.

By Excision, $H_k(S^n - B, B_-^n - B) \cong H_k(S^n, B_-^n)$.

We also have a theorem that homology is
preserved under deformation retract.



Since $S^n - B$ deformation retracts onto B_+^n
and B_-^n deformation retracts to S^{n-1} under this
larger deformation retraction, we gather

$$H_k(B_+^n, S^{n-1}) \cong H_k(S^n, B_-^n)$$

|||

$$H_k(S^n - \text{int}(B_-^n), B_-^n - \text{int}(B_-^n))$$



If $k \geq 1, n \geq 1$: $H_k(S^n) \cong H_k(S^n, B_-^n) \cong H_k(B_+^n, S^{n-1}) \cong \begin{cases} H_k(S^{n-1}) & (k \geq 1) \\ 0 & (k=0) \end{cases}$

Let $k > n \geq 1$:

$$H_k(S^n) \cong H_{k-1}(S^{n-1}) \cong \dots \cong H_{k-n}(S^0) \cong 0.$$

Let $0 \leq k < n$: $H_k(S^n) \cong H_{k-1}(S^{n-1}) \cong \dots \cong H_0(S^{n-k+1}) \cong H_1(S^{n-k+1}, B_-^{n-k+1}) \cong H_1(B_+^{n-k+1}, S^{n-k}) \cong 0.$

Let $1 \leq k = n$

$$H_n(S^n) \cong H_{n-1}(S^{n-1}) \cong \dots \cong H_1(S^1) \cong H_1(S^1, B_-^1) \cong H_1(B_+^1, S^0) \cong \mathbb{R}.$$

If $k \geq 1, n = 0$

$$H_k(S^0) = 0.$$

If $k = 0, n \geq 1$

$$H_0(S^n) = \mathbb{R}.$$

If $k = 0, n = 0$

$$H_0(S^0) = \mathbb{R} \oplus \mathbb{R}.$$

Interior Products

& The Lie Derivative, ^{Other Results} of Differential Geometry. 1

Def/ Let X be a (smooth) vector field on a manifold M . The interior product associated w/ X is the map $i_X : \Omega^p(M) \rightarrow \Omega^{p-1}(M)$ defined by the property that

$$i_X(\omega)(X_1, X_2, \dots, X_{p-1}) = \omega(X, X_1, X_2, \dots, X_{p-1}).$$

Proposition (i_X is an antiderivation of deg -1)

If α is a p -form and β is a q -form,

then
$$i_X(\alpha \wedge \beta) = i_X(\alpha) \wedge \beta + (-1)^p \alpha \wedge i_X(\beta).$$

Recall: Let M be a manifold, and let X be a vector field on M . Fix a point $p \in M$.

There exists $\varepsilon > 0$ and a neighborhood U of p together w/ a family $\theta_t : U \rightarrow M$ of smooth maps which are diffeomorphisms onto their images, where $t \in (-\varepsilon, \varepsilon)$,

satisfying $\theta_0 = \text{the inclusion } U \hookrightarrow M$,

$$\frac{d}{dt} \theta_t(p) = X_{\theta_t(p)}, \quad \text{and there exists}$$

$$W \subset U \text{ s.t. } \theta_t(W) \subset U \quad \text{and} \quad \theta_{-t}|_{\theta_t(W)} = (\theta_t|_W)^{-1}.$$

This construction is called the flow of X near p .

Def/ Let M be a mfld. Let X, Y be v.f.'s on M .

Let $p \in M$. The Lie derivative of Y w.r.t. X

at the point p is defined to be

$$(\mathcal{L}_X Y)_p := \lim_{t \rightarrow 0} \frac{T_{\theta_t(p)}(\theta_{-t})(Y_{\theta_t(p)}) - Y_p}{t} = \left. \frac{d}{dt} \right|_{t=0} T_{\theta_t(p)}(\theta_t)(Y_{\theta_t(p)}).$$

which is an elt of $T_p M$. ^{when it exists} Here θ is the flow of X near p .

Lemma/ If X and Y are smooth vector fields on a smooth mfld M , then for every $p \in M$ the Lie derivative $(\mathcal{L}_X Y)_p$ exists and the resulting v.f. is a Smooth.

Theorem / Let M be a mfld, and let
 X and Y be v.f.'s on M .
Then $\mathcal{L}_X Y = [X, Y]$.

Theorem / Let M be a mfld, and let X and Y
be v.f.'s on M , w/ corresponding flows θ and γ .

By shrinking ε and U we may assume

$\theta_s \circ \gamma_t(q)$ and $\gamma_s \circ \theta_t(q)$ make sense

for all $s, t \in (-\varepsilon, \varepsilon)$ and $q \in U$.

If these two values are equal for all parameters,
including p , we say the flows θ and γ commute.

The theorem is that

$[X, Y] = 0$ if and only if θ and γ commute.

Def & Theorem / There is a notion of the
Lie derivative of a differential
form ω w.r.t. a vector field X on a mfld M .
Known as Elie Cartan's "magic" formula, this can be calculated as
$$\mathcal{L}_X \omega = i_X(d\omega) + d(i_X \omega)$$

Gaussian Curvature

There is a def in problem 3 of Fall 2010.

Thm/ (Gauss-Bonnet)

w/out boundary

If M is a compact ^{oriented} embedded surface^V in $\mathbb{R}^3$,

then
$$\int_M K = 2\pi \chi(M)$$

where K is the Gaussian curvature and $\chi(M)$ is the Euler characteristic.

Thm/ A compact embedded surface w/out boundary in $\mathbb{R}^3$ must have a point of positive

curvature. (Not true if 'embedded' replaced by 'immersed'.)

Orientations (adapted from Bonahon)

Def/ Let M be an m -dim mfd w/ atlas

$\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$. The atlas $\mathcal{A}$ is

oriented if for all $i, j \in I$ and for each $y \in \varphi_i(U_i \cap U_j)$ we have $\det T_y(\varphi_j \circ \varphi_i^{-1}) > 0$.

Def/ Two oriented atlases $\mathcal{A}, \mathcal{A}'$ are

compatible ^{as oriented atlases} if $\mathcal{A} \cup \mathcal{A}'$ is oriented

(in particular, $\mathcal{A}$ and $\mathcal{A}'$ are compatible as atlases).

Rmk/ • Compatibility as oriented atlases is an equivalence relation among oriented atlases.

Def/ An orientation for the mfd M is a choice of equivalence class of oriented atlases.

Lemma/ Every oriented atlas is contained in a unique maximal oriented atlas.

Rmk/ • Here, 'maximal' is w.r.t. the obvious partial ordering by inclusion on the collection of oriented atlases.

- An orientation for M is the same as a choice of maximal oriented atlas.
 - In practice, an oriented atlas for M defines a unique orientation.
-

Def/ Let M be a mfd of dim m .

A volume form on M is a form $\omega \in \Omega^m(M)$

of top-degree m s.t. $\forall x \in M$,

$$\omega_x \neq 0 \in \underbrace{\text{Alt}^m(T_x M)}_{\cong \mathbb{R}}$$

Thm/ M admits an orientation
if and only if M admits a volume form.

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Def/ Two volume forms ω, ω' are
equivalent, denoted $\omega \sim \omega'$, if there exists
a real-valued function $f: M \rightarrow \mathbb{R}$ s.t.
$$\begin{cases} \omega' = f\omega \\ f > 0 \end{cases}$$

Cor/ There is a bijection
$$\left\{ \begin{array}{c} \text{orientations} \\ \text{on } M \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{volume} \\ \text{forms} \end{array} \right\} / \sim$$

Cor/ If M is connected, it admits
either 0 or $\mathbb{Z}$ orientations.

Def/ Let X and Y be two vector fields on an
 m -dim mfd M . The Lie bracket $[X, Y]$ of

X and Y is a vector field on M

defined by $[X, Y]_p = X_p(Y(f)) - Y_p(X(f))$.

Here we are viewing $Y(f)$ and $X(f)$ as functions $M \rightarrow \mathbb{R}$ and f is also a function $M \rightarrow \mathbb{R}$.

Prop / $[fX, gY] =$ $f g [X, Y]$
 $f(Xg)Y - g(Yf)X +$

Fact / If X, Y are vector fields on a mfld (of dim 2). Then for fixed $p \in M$, there is a local coordinate system around p s.t. $X = \partial_x$ and $Y = \partial_y$ IF AND ONLY IF

$$[X, Y] = 0.$$

The obvious generalization to higher dimensions is also true.
