Analysis Qualifying Exam

Spring, 1999

- In order to pass, you must do well on both the Real and Complex Analysis parts—high performance on one portion does not compensate for low performance on the other.
- Start each problem on a fresh sheet of paper.

REAL ANALYSIS. **Do only three of the following four problems.**

1. Suppose f_n, where $n = 1, 2, \ldots$, and f are nonnegative functions on a measure space $(X, \mathcal{M}, \mu)$ with $f_n \to f$ a.e. and $\int_X f_n \, d\mu \to \int_X f \, d\mu$. Show that $\int_E f_n \, d\mu \to \int_E f \, d\mu$ for every measurable E. (Hint: Use Fatou's Lemma.)

2. Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces and $E \in \mathcal{M} \otimes \mathcal{N}$ (the product σ-algebra in $X \times Y$). Show that every section $E_x = \{y \in Y : (x, y) \in E\}$ is measurable.

3. Let A denote the set of all $f \in C[0,1]$ such that f is monotonic on some open subinterval of $[0,1]$. Show that A is meager (that is, of the first category) in $C[0,1]$ in the topology of uniform convergence.

4. (a) Show that the class of all step functions, of form $\sum_{j \le n} c_j \chi_{(a_j, b_j)}$ with a_j, b_j finite, is dense in $L^1(\mu)$ where μ is the Lebesgue measure on $\mathbb{R}$. (Hint: Why is the corresponding statement true for simple functions?)

   (b) Suppose $f \in L^1(\mu)$. Show that $\lim_{h \to 0} \int |f(x+h) - f(x)| \, dx = 0$. (Hint: Use (a).)

COMPLEX ANALYSIS. **Do all four problems.**

5. Suppose that f is analytic on $\mathbb{C}$ and that f is a homeomorphism of $\mathbb{C}$ onto a set U.

   (a) Show that f has a non-essential singularity at ∞.

   (b) Deduce that f must be of the form $f(z) = az + b$ for some $a \neq 0$ and that $U = \mathbb{C}$.

6. (a) Suppose that f is analytic on the open unit disc $|z| < 1$ and there is a constant M such that $|f^{(k)}(0)| \le k^2 M^k$ for all $k \ge 1$. Show that f can be extended to be analytic on $\mathbb{C}$.

   (b) Suppose that f is analytic on the open unit disc $|z| < 1$ and there is a constant $M > 1$ such that $|f(1/k)| \le M^{-k}$ for $k \ge 2$. Show that f is identically zero.