SPRING 1998 ANALYSIS QUALIFYING EXAM
MONDAY, MAY 4, 1998

DIRECTIONS. Do any seven of the following eight problems, **using the paper and pens provided**. Start each problem on a *fresh* sheet of paper. When you have completed the exam, be sure your name is printed on each page; sign the envelope, and return the exam papers in the envelope. You may keep this printed page.

Problem 1. Suppose $f \in L^1(d\mu)$. Prove: for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each measurable set A with $\mu(A) < \delta$, there holds

$$\left| \int_A f \, d\mu \right| < \varepsilon.$$

Problem 2. Let f be an entire function which is real on the real axis, not identically zero, and for which $f(0) = 0$. Prove: if f maps the imaginary axis into a straight line, then that straight line must be either the real axis or the imaginary axis.

Problem 3. Suppose $\{f_n\}$ is a sequence of continuously differentiable functions on $[0,1]$ which converges in the L^1 sense to 0, and whose derivatives $\{f_n'\}$ also converge to 0 in the L^1 sense. Prove: $\{f_n\}$ converges to zero uniformly.

Problem 4. Suppose D is the open unit disk in $\mathbb{C}$, and $f : D \to D$ satisfies $f(1/2) = 1/2$. Show that $|f'(1/2)| \leq 3/4$.

Problem 5. Let $(X, \mathcal{T})$ be a topological space which has the property that every closed set F is the intersection of a *countable* family of open sets. Prove: any finite measure μ on the Borel field of $(X, \mathcal{T})$ is *regular*: for each Borel set E and each $\varepsilon > 0$, there exist an open set $G \supset E$ and a closed set $F \subset E$ such that $\mu(G \backslash F) < \varepsilon$. (Hint: consider the collection of Borel sets E for which this condition is true.)

Problem 6. Let D be the open unit disk in $\mathbb{C}$, and let $f : D \to D$ be analytic with $f(0) = 0$. Suppose

$$|f(z)| \geq \frac{1}{6} \quad \text{for all} \quad |z| = \frac{1}{4}.$$

Show that f assumes every value in the disk $|w| < \dfrac{1}{6}$.

Problem 7. Let $g : [0,1] \to \mathbb{R}$ be Lebesgue measurable, and suppose $f(x,y) := g(x) - g(y)$ is Lebesgue integrable on $[0,1] \times [0,1]$. Prove: g is Lebesgue integrable.

Problem 8. Evaluate: $\int_0^\infty \dfrac{\sqrt{x}}{1+x^3}\, dx$.

Q4: ... f analytic --- Show that $|f'(\frac{1}{2})| \leq 1$

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