REAL AND COMPLEX ANALYSIS QUALIFYING EXAM

SPRING 1997

Directions: Do any **seven** of the following eight problems.

Problem 1 Prove: if $n \geq 2$ is an integer, then

$$\int_0^\infty \frac{\mathrm{d}x}{1+x^n} = \frac{x/n}{\sin(\pi/n)}$$

Problem 2 Suppose Ω is an open connected region of the complex plane and f is a non-constant analytic function on $\overline{\Omega}$. Prove: if $|f(z)| \equiv 1$ on the boundary of Ω, then $f(z)$ has at least one zero in Ω.

Problem 3 Formally, we have that

$$\begin{aligned}\frac{(-1)^n n!}{t^{n+1}} &= \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\frac{1}{t}\right) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^\infty e^{-tx}\,\mathrm{d}x \\ &= \int_0^\infty \frac{\partial^n}{\partial t^n} e^{-tx}\,\mathrm{d}x = \int_0^\infty (-1)^n x^n e^{-tx}\,\mathrm{d}x\end{aligned}$$

so that on setting $t = 1$ we obtain

$$\int_0^\infty x^n e^{-x}\,\mathrm{d}x = n!$$

Justify the calculation.

Problem 4 Let $X = C[0,1]$ be the space of all bounded continuous functions from $[0,1]$ to $\mathbb{R}$ with the sup-norm distance,

$$d(f,g) = \sup_{0\leq t\leq 1} |f(t) - g(t)|$$

You may assume that (X,d) is complete. Let $F : X \to X$ be a strict contraction, i.e., a function such that there exists $k < 1$ with

$$d(Fx, Fy) \leq k d(x,y) \quad \text{for all} \quad x, y \in X$$

Let I denote the identity operator on X, prove:

- $I + F$ is a 1-1 mapping of X onto X
- $(I+F)^{-1}$ is continuous

Problem 5 Let $K : [0,1]\times[0,1] \to \mathbb{R}$ be continuous, and let $\mathcal{F}$ be the family of all functions f on $[0,1]$ of the form

$$f(x) = \int_0^1 g(y)K(x,y)\,\mathrm{d}y$$

Problem 6 Show that for each $\varepsilon > 0$ the function

$$f(z) = \sin z + \frac{1}{z}$$

has infinitely many zeros in the strip $|\Im z| < \varepsilon$.

Problem 7 Determine the order of the entire function

$$f(z) = \prod_{n=1}^\infty \left(1 + \frac{z}{n^2}\right)$$

(Recall that the *order* of an entire function f is

$$\lim_{r\to\infty} \frac{\log\log M(r)}{r}$$

where $M(r) = \max_{|z|=r} |f(z)|$.)

Problem 8 Prove: if A and B are Lebesgue-measurable subsets of $\mathbb{R}$ with positive Lebesgue measure, then the set

$$A + B = \{a + b : a \in A, b \in B\}$$

has non-empty interior. (Hint: consider the convolution of the characteristic functions of A and B.)