Ma525a Qualifying Exam
Choose four of the following five questions.

1. Let f_n, f be real or complex valued measurable functions on the measure space $(X, \mathcal{M}, \mu)$. Suppose that f_n converges to f in measure; that is,

$$\forall \epsilon > 0 \quad \lim_{n\to\infty} \mu\{x : |f_n(x) - f(x)| \geq \epsilon\} = 0.$$

a) Show that if there is a g such that $|f_n| \leq g$ a.e., then $|f| \leq g$ a.e.

b) Suppose that $\mu(X) < \infty$ and $|f_n| \leq g$ a.e with $\int g d\mu < \infty$. Show that

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu.$$

2. For a function $f : \mathbf{R} \to \mathbf{R}$, recall that

$$\limsup_{y\to x} f(y) = \lim_{\delta\to 0} \sup\{f(y) : |y - x| < \delta\}.$$

We say that f is *upper semi-continuous*, or u.s.c., if

$$\limsup_{y\to x} f(y) \leq f(x).$$

Prove that f is measurable if f is u.s.c.

3. Let ϕ be a measurable complex function on $\mathbf{R}$ satisfying

$$|\phi(x)| = 1 \quad \text{and} \quad \phi(x + y) = \phi(x)\phi(y) \quad \text{for all } x, y \in \mathbf{R}.$$

Prove that ϕ is continuous. (Hint: Show there exists a such that $A = \int_0^a \phi(t)dt \neq 0$ and consider $A^{-1} \int_0^a \phi(x + t)dt$.)

4. For $0 < \alpha < 1$, the sequence of numbers $\xi_n = \alpha^{n/(n+1)}2^{-n}$ satisfies $\xi_n > 2\xi_{n+1}$, $n = 0, 1, \ldots$. Let $B_0 = [0, 1]$. To obtain B_{n+1} given B_n, the union of intervals, remove from the middle of each subinterval of B_n the open interval of length $\xi_n - 2\xi_{n+1}$. Show that $B = \cap_{j=0}^{\infty} B_n$ is a closed, nowhere dense subset of $[0, 1]$ with measure α.

5. Let A be a bounded measurable subset of $\mathbf{R}$. Show that

$$\lim_{n\to\infty} \int_A \cos(nx)dx = 0.$$