Spring 1995

Analysis Qualifying Examination
Tuesday, May 2, 1995

INSTRUCTIONS. Do **any seven** of the following problems; begin each problem on a fresh sheet of paper.

Problem 1. *Let $f(z)$ be analytic in $|z| \le 1$, with $f(0) = a_0 \ne 0$. If $M = \max_{|z|=1} |f(z)|$, show that $f(z) \ne 0$ for*

$$|z| < \frac{|a_0|}{|a_0| + M}.$$

Problem 2. *Let μ be a finite measure on $(X, \mathcal{B})$, suppose $f_n \to f$ a.e. on X, and $\|f_n\|_2 \le M < \infty$ for all n. Show that $f_n \to f$ in L^1.*

Problem 3. *Compute:*

$$\int_{|\xi|=1} \frac{d\xi}{\sqrt{\xi}}.$$

Problem 4. *Suppose μ is a measure on $(X, \mathcal{B})$, and suppose the function $f : \mathbb{R} \times X \to \mathbb{R}$ is differentiable in the L^2 sense, that is, $f(t, \cdot) \in L^2(\mu)$ for all t and $(f(t+h, \cdot) - f(t, \cdot))/h$ converges in $L^2(\mu)$ to a limit $g(t, \cdot)$ as $h \to 0$. Define*

$$\alpha(s,t) = \int_X f(s,x) f(t,x)\, d\mu(x).$$

Prove: $\dfrac{\partial^2 \alpha}{\partial t \partial s}$ exists.

Problem 5. *Show that $f(z) = z^5 + z^3 + 2z + 3$ has only one zero in the first quadrant $x \ge 0, y \ge 0$ (where $z = x + iy$).*

Problem 6. *Investigate the convergence of $\sum_n u_n$, where*

$$u_n = \int_0^1 \frac{x^n}{1-x} \sin(\pi x)\, dx.$$

Problem 7. *If $f(z)$ is analytic for $|z| < 1$ and $f(0) = 0$, prove that*

$$\sum_{n=1}^{\infty} f(z^n)$$

converges in $|z| < 1$ to an analytic function.

Problem 8. *If $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable, prove that its graph*

$$G = \{(x, f(x)) : x \in \mathbb{R}\}$$

has Lebesgue measure zero in $\mathbb{R}^2$. (Hint: first do it for bounded functions.)