Real Analysis Qual

Spring 1994.

② Suppose $[0,1] = \bigsqcup_{i=1}^{\infty} [a_i, b_i]$ w/ $a_i < b_i$ so $[a_i, b_i] \neq \emptyset$ $\forall i$.

Since they are disjoint & countable we can order them so WLOG assume $0 \le \ldots a_i < b_i < a_{i+1} < b_{i+1} \ldots \le 1$. Note that $b_i < a_{i+1}$ since by assumption all the closed intervals are disjoint.

Hence for any i $(b_i, a_{i+1}) \neq \emptyset$ (since $b_i < a_{i+1}$) and $(b_i, a_{i+1}) \cap \left(\bigsqcup^{\infty} [a_i, b_i]\right) = \emptyset$ (since we assumed the intervals were ordered).

But $[a_i, b_i] \subseteq [0,1]$ & since $[0,1]$ is connected it follows $(b_i, a_{i+1}) \subseteq [0,1]$.

But we said $[0,1] = \bigsqcup^{\infty} [a_i, b_i]$ $\Rightarrow\Leftarrow$. Thus $[0,1]$ cannot be written as a countable disjoint union of closed intervals. ▫

④ $f : [1, +\infty) \to [0, \infty)$

Want: $\displaystyle\int_1^{\infty} \frac{f^2(x)}{x^2}\, dx < \infty \Rightarrow \int_1^{\infty} \frac{f(x)}{x^2}\, dx < \infty$.

The proof directly follows from Hölder's inequality for $p = q = 2$.

Explicitly: for any $a, b \in \mathbb{R}^+ \Rightarrow 0 \le (a-b)^2 = a^2 + b^2 - 2ab$

$$\Rightarrow ab \le \frac{a^2}{2} + \frac{b^2}{2}$$

Thus $\displaystyle\left|\frac{f(x)}{x^2}\right| = \left|\frac{f(x)}{x} \cdot \frac{1}{x}\right| \le \frac{1}{2}\left(\frac{f(x)}{x}\right)^2 + \frac{1}{2}\left(\frac{1}{x}\right)^2$

Hence $\displaystyle\int_1^{\infty} \frac{f(x)}{x^2}\, dx \le \int_1^{\infty} \frac{|f(x)|}{x^2}\, dx \le \frac{1}{2}\int_1^{\infty} \frac{f^2(x)}{x^2}\, dx + \frac{1}{2}\int_1^{\infty} \frac{1}{x^2}\, dx < \infty$

since $f(x)/x$ & $\frac{1}{x} \in L^2([1,\infty))$. ▫

⑥ $([0,1], \mathcal{A}, \mu)$ Lebesgue measure space on $[0,1]$

a) $f : [0,1] \to \mathbb{R}$ continuous a.e. $\not\Rightarrow$ $\exists\, g : [0,1] \to \mathbb{R}$ conts. w/ $f = g$ a.e.

Consider $f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ continuous everywhere except at $x=0$.

Then if $\exists\, g = f$ a.e. continuous since $[0,1]$ is compact, g must be bounded. But $\lim_{x\to 0} g(x) = \lim_{x \to 0} f(x) = \infty$. $\Rightarrow\Leftarrow$.

Thus such a g cannot exist.

b) $\exists\, g$ continuous $\ni$ $f = g$ a.e. $\not\Rightarrow$ f is continuous a.e.

Consider $f(x) = \begin{cases} 0 & x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1] \\ 1 & x \in \mathbb{Q} \cap [0,1]. \end{cases}$

Then f is discontinuous everywhere but $g \equiv 0$ is continuous and $g = f$ a.e. since $m(\mathbb{Q}) = 0$.

$\square$

(8) $\{f_n\}$ m'able w/ $\mu(X) < \infty$

a) Suppose $\exists\, g \in L^1(\mu) \ni |f_n(x)| \le g(x)\ \forall x, n. \Rightarrow f_n$ is unif. integ.

Since $g \in L^1$ and $\mu(X) < \infty$ then $\int_X |g(x)|dx < \infty \Rightarrow |g(x)| < \infty$ a.e. $x \in X$

But then $\exists\, M \ni |g(x)| \le M$ a.e. x.

Thus let $R > M$, so $m(\{x ; g(x) \ge R\}) = 0 \Rightarrow \int_{|g(x)| \ge R} |g(x)|dx = 0$

Then since $|f_n(x)| \le g(x) \le |g(x)|\ \forall x$ we have:

$$\{x ; |f_n(x)| \ge R\} \subseteq \{x ; |g(x)| \ge R\} \text{ \& hence:}$$

$$\sum_{n=1}^{\infty} \int_{|f_n| \ge R} |f_n(x)|dx \le \sum_{n=1}^{\infty} \int_{|g(x)| \ge R} |g(x)|\, dx = 0 \qquad \forall n.$$

Thus $\lim_{R \to \infty} \sum_{n=1}^{\infty} \int_{|f_n| \ge R} |f_n(x)|dx = 0$.

b) Suppose $f_n \to f$ pointwise & $\{f_n\}$ are uniformly integrable

$\Rightarrow f \in L^1$ & $\int f_n \to \int f$.

Since $\sum_{n=1}^{\infty} \int_{|f_n| \ge R} |f_n(x)|d\mu \ge \int_{|f_n| \ge R} |f_n(x)|\, d\mu$ for any n then

$\forall \epsilon\ \exists R \ni \int |f_n(x)|d\mu = \int_{|f_n| \ge R} |f_n(x)|d\mu + \int_{|f_n| < R} |f_n(x)|d\mu < \epsilon + R \cdot \mu(X) < \infty\ \forall n.$

Thus $|f_n(x)| \le M$ a.e. $x\ \forall n$. Since $\int M d\mu = \mu(X) \cdot M < \infty$ and $f_n \to f$ then by L.D.C.T. we have $f \in L^1$ & $\int f_n \to \int f$. ∎