REAL AND COMPLEX ANALYSIS QUALIFYING EXAM

SPRING 1994

Problem 1 Evaluate $\int_0^\infty \frac{\log x}{1+x^2}\, dx$

Problem 2 Show that $[0, 1]$ cannot be written as the countably infinite union of disjoint nonempty closed intervals.

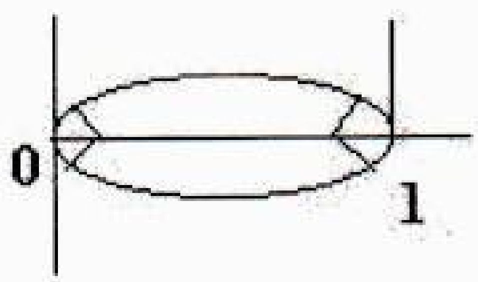
Problem 3 Let $f : D \to \mathbb{C}$ be analytic such that $\Re f(z) > 0$ for all z. Prove

$$|f(z)| \quad |f(0)|\frac{1+|z|}{1 \quad |z|}$$

Problem 4 Let $f : [1, +\infty) \to [0, +\infty)$ be Lebesgue measurable. Prove:

$$\int_1^\infty \frac{f(x)^2}{x^2} < +\infty \quad \Rightarrow \quad \int_1^\infty \frac{f(x)}{x^2}\, dx < +\infty$$

Problem 5 Map the region between the circular arcs (in the figure below) conformally to the unit disk. (Note that the top and bottom acrs intersect at right angles.)



Problem 6 Let $([0, 1], \mathcal{A}, \quad)$ denote the Lebesgue measure space on $[0, 1]$. Give examples to show that for $f : [0, 1] \to \mathbb{R}$ the condition "f is continuous a.e." neither implies, nor is imlied by, the condition "there exists a continuous function $g : [0, 1] \to \mathbb{R}$ such that $f = g$ a.e."

Problem 7 An entire function is said to have **finite order** if there exists $c > 0$ such that $|f(z)| \quad \exp(|z|^c)$ for all $|z|$ sufficiently large; the **order** of f is the infimum of all such $c > 0$. Prove that the following function is entire and has order $1/2$:

$$f(z) = \prod_{k=1}^{\infty} 1 \quad \frac{z}{k^2}$$

Problem 8 Let $\{f_n\}$ be a sequence of measurable functions on some measure space $(X, \mathcal{A}, \quad)$ with $\quad (X) < \infty$. We say the sequence is **uniformly integrable** if

$$\lim_{R\to\infty} \sum_n \int_{|f_n| \quad R} |f_n|\, d \quad = 0$$

(a) Show that if there exists $g \in L^1(\quad)$ such that $|f_n(x)| \quad g(x)$ for all x, n there the $\{f_n\}$ are uniformly integrable.

(b) Prove that if $f_n \to f$ pointwise and the $\{f_n\}$ are uniformly integrable then $f \in L^1(\quad)$ and

$$\lim_n \int f_n\, d \quad = \int f\, d$$