Real Analysis Qual

Spring 1992

① $f_n \in L^1(\mu)$; $f_n \to f$ a.e.

$(\Rightarrow)$ assume $f_n \to f$ in $L^1(\mu)$. Then $\int |f_n - f| d\mu \to 0$.

So then $\left| \int |f_n| - |f| d\mu \right| \le \int \big| |f_n| - |f| \big| d\mu \le \int |f_n - f| d\mu \to 0$

So $\int |f_n| \to \int |f|$

$(\Leftarrow)$ Suppose $\int |f_n| \to \int |f|$.

Then by fatou $\int |f| \le \liminf \int |f_n|$. so:

$2\int |f| - \limsup_n \int |f_n - f| d\mu$ (since $\liminf (a+b) \le \limsup a + \limsup b$)

$\ge \limsup_n \left[\int |f| + |f_n| - |f_n - f| \right]$

$\ge \liminf_n \left[\int |f| + |f_n| - |f_n - f| \right]$] since $\int |f_n| \to \int |f|$ & by fatou.

$\ge 2\int |f| - \int \liminf |f_n - f|$

$= 2\int |f|$

$\Rightarrow \limsup \int |f_n - f| d\mu \le 0$

now $\int |f_n - f| d\mu \ge 0 \Rightarrow \liminf \int |f_n - f| d\mu \ge 0$

$\Rightarrow \lim_n \int |f_n - f| d\mu = 0$. □

(2) $f_n : \mathbb{R} \to [0,1]$ Lebesgue m'able $\forall n$ $\ni$ $\lim_{n\to\infty} \int_0^1 |f_n(x)|\,dx = 0$

Then $\forall \epsilon > 0\ \exists N \ni \forall n \geq N$ $\int_0^1 |f_n(x)|dx < \epsilon$

Now $\int_0^1 |f_n(x)|dx \leq \max_{x\in[0,1]} |f_n(x)| < \epsilon$ since f_n is m'able then $\{x ; f_n(x) = \infty\}$ has measure zero, hence we can consider such a maximum existing for a.e. $x \in [0,1]$.

$\Rightarrow |f_n(x)| < \epsilon$ a.e. $x \in [0,1]$ for $\forall n \geq N$.

Thus let $\epsilon = \frac{1}{m}$ for $m \in \mathbb{N}$.

Then for each m $\exists N_m \ni |f_{N_m}(x)| \leq \frac{1}{m}$ a.e. x

Thus $|f_{N_m}(x)| \to 0$ for a.e. x.

$\square$

(3) Prove the Lebesgue measure is translation invariant.

First suppose A is an open interval. Then for any $x \in \mathbb{R}$ we have:

$x + A = x + (a,b) = \underbrace{(a+x, b+x)}_{\text{since again is open set.}} \in \mathcal{M}$ for any A open.

Now $\lambda(x+A) = \lambda(a+x, b+x) = b+x-a-x = b-a = \lambda(A)$

Since any open set is the union of open sets this holds for all open sets & hence for complements. Thus it holds for $\mathcal{B}_{\mathbb{R}}$ since it is the σ-algebra generated by open sets in $\mathbb{R}$. Now given $A \in \mathcal{L}$, then $A = B \cup N$ where B is Borel & N is Lebesgue-null.

Since $A + x = (B+x) \cup (N+x)$ where $B+x \in \mathcal{B}_{\mathbb{R}}$ & $\lambda(N+x) = 0$ then $A+x \in \mathcal{L}$. Clearly $\lambda(A+x) = \lambda((B+x) \sqcup (N+x)\setminus(B+x))$

$= \lambda(B+x) + \lambda(N+x \setminus B+x) = \underset{\uparrow \text{ by above}}{\lambda(B)} + \underset{\uparrow \text{ by completeness of } \lambda.}{0} = \lambda(B) + \lambda(N\setminus B) = \lambda(A)$

(4) $f: \mathbb{R} \to \mathbb{R}$ w/ $f(x_0) \leq \liminf\limits_{n \to \infty} f(x_n) \quad \forall x_n \to x_0$

so $\quad f(x_0) \leq \inf \{ \liminf\limits_{n \to \infty} f(x_n) \, ; \, x_n \to x_0 \}$.

Hence $\quad f(x_0) \leq \liminf\limits_{x \to x_0} f(x)$.

So then $\quad \liminf\limits_{x \to x_0} f(x) = \lim\limits_{\delta \to 0} \left(\inf\limits_{0 < |x - x_0| < \delta} f(x) \right) \geq f(x_0)$

$\Rightarrow \forall \epsilon > 0 \ \exists \delta \ni \left| \liminf\limits_{x \to x_0} f(x) - \inf\limits_{|x - x_0| < \delta} f(x) \right| < \epsilon$

So then $\quad \liminf\limits_{x \to x_0} f(x) - \epsilon \leq \inf\limits_{|x - x_0| < \delta} f(x)$

hence for any $x \in B_\delta(x_0)$ we have:

$$f(x_0) - \epsilon \leq \liminf_{x \to x_0} f(x) - \epsilon \leq f(x)$$

So then to show f is Borel m'able we must show $f^{-1}((a, \infty)) \in \mathcal{B}_{\mathbb{R}} \ \forall a \in \mathbb{R}$

so then fix $a \in \mathbb{R}$ and consider $\{x : f(x) > a\} = E$.

Then if $x_0 \in E$ we have that $f(x_0) > a$.

So then $\quad f(x) \geq f(x_0) - \epsilon > a - \epsilon \quad \forall x \in (x_0 - \delta, x_0 + \delta)$

Since ϵ is arbitrary $\Rightarrow f(x) > a \ \forall x \in B_\delta(x_0)$.

So $f(B_\delta(x_0)) \subseteq (a, \infty)$. Hence $B_\delta(x_0) \subseteq E$ so E is open.

But open sets are Borel m'able by definition so $E \in \mathcal{B}_{\mathbb{R}}$. Thus f is Borel m'able.

$\square$

(3) $\varphi(p) = \int_0^\infty x^p e^{-x} dx \quad p \geq 0$

<u>Well defined</u>: $|\varphi(p)| < \infty \ \forall p \geq 0 \iff \left| \int_0^\infty x^p e^{-x} dx \right| < \infty \ \forall p \geq 0$.

it suffices to show $x^p e^{-x} \in L^1 \ \forall p \in (0,\infty)$

Now $\forall p \in (0,\infty) \ \exists a \ni \forall x > a, \quad x^p \leq e^{x/2}$

Thus $\int_0^\infty x^p e^{-x} dx = \int_0^a x^p e^{-x} dx + \int_a^\infty x^p e^{-x} dx$

$\leq a^p \int_0^a e^{-x} dx + \int_0^a e^{-x/2} dx < \infty$

So $x^p e^{-x} \in L^1$ & hence $|\varphi(p)| < \infty$ & so well defined

<u>Differentiable</u>:

Now we know that for $\varphi(p) = \int_a^b f(p,x) dx$ w/ $f(p,x)$ integrable $\forall p$

and where $\frac{\partial}{\partial p} f(p,x) \ \exists \ \forall x$ & $\left|\frac{\partial}{\partial p} f(p,x)\right| \leq g(x) \in L^1$

$\Rightarrow \varphi(p)$ diff'l.

So by well definedness $\Rightarrow f(p,x) = x^p e^{-x}$ integrable $\forall p$.

Clearly $\frac{\partial f}{\partial p} = x^p e^{-x} \ln x$ exists $\forall x \in [0,\infty)$

& $|x^p e^{-x} \ln x| \leq$