Spring 92

REAL ANALYSIS QUALIFYING EXAM

SPRING 1992

Problem 1 Let (X, Σ, μ) be a measure space and $\{f_n\}$ a sequence in $L^1(\,d\mu)$ which converges a.e. to $f \in L^1(\,d\mu)$. Prove: $f_n \to f$ in $L^1(\,d\mu)$ iff $\int |f_n|\,d\mu \to \int |f|\,d\mu$.
Hint: Apply Fatou's lemma to $|f| + |f_n| - |f - f_n|$.

Problem 2 Let $\{f_n\}$ be a sequence of Lebesgue-measurable real-valued functions on $[0, 1]$ such that

$$\lim_{n\to\infty} \int_0^1 |f_n(x)|\,dx = 0$$

Prove: there exists a subsequence of $\{f_n\}$ such that $\{f_{n_i}(x)\}$ converges to 0 for a.e. x.

Problem 3 Prove that Lebesgue measure λ on $\mathbb{R}$ is translation-invariant: if A is a Lebesgue-measurable subset of $\mathbb{R}$, then for each $u \in \mathbb{R}$, $u + A$ is also Lebesgue-measurable and $\lambda(u + A) = \lambda(A)$.

Problem 4 A function $f : \mathbb{R} \to \mathbb{R}$ is said to be **lower semi-continuous** provided

$$f(x) \le \liminf_{n\to\infty} f(x_n)$$

whenever $\lim_n x_n = x$. Show that every lower semi-continuous function is Borel measurable.

Problem 5 Show that the function φ defined by

$$\varphi(p) = \int_0^\infty x^p e^{-x}\,dx \quad (p \ge 0)$$

is well-defined and differentiable on $(0, \infty)$.