REAL ANALYSIS GRADUATE EXAM

Spring 2025

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Assume that $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable almost everywhere with respect to the Lebesgue measure. Prove that the derivative f' is Lebesgue measurable.

2. Evaluate the limit

$$\lim_{n\to\infty} \int_0^1 \frac{n}{1+n^2x^2+n^6x^8}\,dx.$$

Make sure to justify all your steps.

3. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Suppose that $f_n \in \mathscr{L}^1(\mu)$ is a sequence of functions with the property that for every $\varepsilon > 0$ there exists a $\delta > 0$ so that

$$E \in \mathcal{M} \text{ and } \mu(E) < \delta \implies \sup_n \int_E |f_n|\,d\mu < \varepsilon. \tag{0.1}$$

Suppose in addition that there exists f with $f_n \to f$ μ-a.e. Prove that $f_n \to f$ in $\mathscr{L}^1(\mu)$.

4. For $x \in \mathbb{R}^n$ and $A \subseteq \mathbb{R}^n$, denote $\operatorname{dist}(x, A) = \inf_{y\in A} |x-y|$.
(i) Let $A \subset \mathbb{R}^n$ be a compact subset. For $k \in \mathbb{N}$, let $A_k = \{x \in \mathbb{R}^n : \operatorname{dist}(x, A) \le \frac{1}{k}\}$. Prove that $\lim_{k\to\infty} m(A_k) = m(A)$, where m is the Lebesgue measure.
(ii) Show that this is not true if A instead of compact is only Lebesgue measurable. (Hint: Let $n = 1$.)