# Qualifying Exam: Real Analysis

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Spring 2025

1. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable almost everywhere with respect to the Lebesgue measure, then $f'$ is Lebesgue-measurable.

   *Solution.* The derivative $f'$ is defined almost everywhere as the pointwise limit of measurable functions:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{n \to \infty} n\left(f\left(x + \tfrac{1}{n}\right) - f(x)\right).$$

2. Compute

$$\lim_{n \to \infty} \int_0^1 \frac{n}{1 + n^2 x^2 + n^6 x^8} \, \mathrm{d}x.$$

   *Solution.* The trick is to apply the change of variables $y = nx$, which yields

$$\lim_{n \to \infty} \int_0^1 \frac{1}{1 + y^2 + n^{-2} y^8} \, \mathrm{d}y.$$

   With $1/(1 + y^2)$ as the dominating function, the dominated convergence theorem yields the answer

$$\int_0^1 \lim_{n \to \infty} \frac{1}{1 + y^2 + n^{-2} y^8} \, \mathrm{d}y = \int_0^1 \frac{1}{1 + y^2} \, \mathrm{d}y = \arctan(1) - \arctan(0) = \frac{\pi}{4}.$$

3. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Suppose that $\{f_n\}_{n \geq 1} \subseteq L^1(\mu)$ is uniformly integrable: for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\forall E \in \mathcal{M} \quad \left(\mu(E) < \delta \implies \sup_n \int_E |f_n| \, \mathrm{d}\mu < \varepsilon\right).$$

   Prove that if $f_n$ converges almost everywhere to $f$, then $f_n$ converges in $L^1$ to $f$.

   *Solution.* Let $\varepsilon > 0$. The uniform integrability of $\{f_n\}_{n \geq 1}$ yields $\delta > 0$ such that

$$\forall E \in \mathcal{M} \quad \left(\mu(E) < \delta \implies \sup_n \int_E |f_n| \, \mathrm{d}\mu < \frac{\varepsilon}{3}\right).$$

   Moreover, by Fatou's lemma,

$$\forall E \in \mathcal{M} \quad \left(\mu(E) < \delta \implies \int_E |f| \, \mathrm{d}\mu \leq \liminf_{n \to \infty} \int_E |f_n| \, \mathrm{d}\mu \leq \sup_n \int_E |f_n| \, \mathrm{d}\mu < \frac{\varepsilon}{3}\right).$$

   Egorov's theorem yields $E \in \mathcal{M}$ such that $\mu(E) < \delta$ and $f_n$ converges uniformly to $f$ on $X \setminus E$. Then, uniform convergence yields $N$ such that for all $n \geq N$,

$$\int_{X \setminus E} |f_n - f| \, \mathrm{d}\mu < \frac{\varepsilon}{3}.$$

Putting all three pieces together yields convergence in $L^1$:

$$\begin{aligned}\lim_{n\to\infty}\int_X |f_n - f|\,\mathrm{d}\mu &\le \lim_{n\to\infty}\int_E |f_n| + |f|\,\mathrm{d}\mu + \lim_{n\to\infty}\int_{X\setminus E} |f_n - f|\,\mathrm{d}\mu \\ &\le \sup_n \int_E |f_n|\,\mathrm{d}\mu + \int_E |f|\,\mathrm{d}\mu + \lim_{n\to\infty}\int_{X\setminus E} |f_n - f|\,\mathrm{d}\mu \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

*Remark.* See the Vitali convergence theorem.

4. Let $A$ be a compact subset of $\mathbb{R}^n$, and let $A_k = \{x \in \mathbb{R}^n : d(x, A) \le \frac{1}{k}\}$, where $d(x, A) = \inf_{y\in A} d(x, y)$. Prove that

$$\lim_{k\to\infty} m(A_k) = m(A).$$

*Solution.* Compact subsets of $\mathbb{R}^n$ are closed and bounded. Since $A$ is bounded, so is $A_1$, and thus $m(A_1) < \infty$. Continuity from above implies that

$$\lim_{k\to\infty} m(A_k) = m\left(\bigcap_{k=1}^{\infty} A_k\right) = m(\{x \in \mathbb{R}^n : d(x, A) = 0\}) = m(\mathrm{cl}(A)) = m(A).$$

Give an example of a measurable subset $A \subseteq \mathbb{R}^n$ for which

$$\lim_{k\to\infty} m(A_k) \ne m(A).$$

*Solution.* Let $A = \mathbb{Q}^n$. Then, $m(A_k) = m(\mathbb{R}^n) = \infty$ for all $k$, but $m(A) = 0$.