Qualifying Exam: Real Analysis

Unofficial solutions by Alex Fu

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1. Let $(f_n)_{n\geq 1}$ be a sequence of C^1 functions from $\mathbb{R}^{\geq 0}$ to $\mathbb{R}$ such that $f_n(0) = 0$ for every n and

$$\lim_{n\to\infty} \int_0^\infty |f_n'(x)|^2 \,\mathrm{d}x = 0.$$

Find a proof or counterexample to the following statement:

$$\lim_{n\to\infty} \sup_{x\geq 0} |f_n(x)| = 0.$$

Solution. We provide the following counterexample. For any $f_n \in \mathrm{C}^1$ with nonnegative derivative,

$$\sup_{x\geq 0} |f_n(x)| = \sup_{x\geq 0} f_n(x) = \sup_{x\geq 0} \int_0^x f_n'(t)\,\mathrm{d}t = \int_0^\infty f_n'(t)\,\mathrm{d}t.$$

Thus, we seek a sequence of nonnegative continuous functions $(g_n)_{n\geq 1} = (f_n')_{n\geq 1}$ that converges to 0 in L^2 but not in L^1. The function $1/x$ belongs to $L^2([1,\infty)) \setminus L^1([1,\infty))$ and hence inspires $g_n(x) = 1/(x+n)$:

$$\int_0^\infty g_n(x)^2\,\mathrm{d}x = \frac{1}{n} \to 0,$$

$$\int_0^\infty g_n(x)\,\mathrm{d}x = \infty \quad \text{for all } n.$$

Then, our counterexample is

$$f_n(x) = \ln(x+n) - \ln n.$$

2. Let $(X, \mathscr{A}, \mu)$ be a complete measure space, let f be a nonnegative integrable function on X, and put $b(t) = \mu(\{x \in X : f(x) \geq t\})$. Show that

$$\int_X f\,\mathrm{d}\mu = \int_0^\infty b(t)\,\mathrm{d}t.$$

Solution. Let $E = \{x \in X : f(x) > 0\}$. Since $E = \bigcup_{n=1}^\infty \{x \in X : f(x) > \frac{1}{n}\}$ is σ-finite, Tonelli's theorem implies that

$$\int_E f(x)\,\mathrm{d}\mu = \int_E \int_0^\infty \mathbb{1}\{f(x) \geq t\}\,\mathrm{d}t\,\mathrm{d}\mu = \int_0^\infty \int_E \mathbb{1}\{f(x) \geq t\}\,\mathrm{d}\mu\,\mathrm{d}t = \int_0^\infty \mu(\{x \in E : f(x) \geq t\})\,\mathrm{d}t.$$

To conclude, note that $\int_X f\,\mathrm{d}\mu = \int_E f\,\mathrm{d}\mu$ and $b(t) = \mu(\{x \in E : f(x) \geq t\})$ for all $t > 0$.

3. Let X be a compact metric space, and let μ be a finite Borel measure on X such that $\mu(\{x\}) = 0$ for every $x \in X$. Prove that for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\mu(E) < \varepsilon$ whenever E is a Borel subset of X with diameter less than δ.

   Solution. The inclusion of each set with diameter less than d in an open ball of radius $d/2$ allows reduction to the case of open balls. For each $x \in X$, continuity from above implies that

$$\lim_{r\to 0}\mu(\mathrm{Ball}(x,r)) = \mu\left(\bigcap_{k=1}^{\infty}\mathrm{Ball}\left(x,\frac{1}{k}\right)\right) = \mu(\{x\}) = 0,$$

   which yields $r_x > 0$ such that $\mu(\mathrm{Ball}(x,2r_x)) < \varepsilon$. Compactness yields a finite subcover $\{\mathrm{Ball}(x_i,2r_{x_i})\}_{i=1}^{n}$ for X. Then, every open ball with radius less than $\delta = \min\{r_{x_1},\dots,r_{x_n}\} > 0$ satisfies, for some $i \in \{1,\dots,n\}$,

$$\mathrm{Ball}(x,r) \subseteq \mathrm{Ball}(x,\delta) \subseteq \mathrm{Ball}(x_i,2r_{x_i}).$$

   Therefore, $\mu(B) < \varepsilon$ whenever B is an open ball with diameter less than 2δ.

4. Let $f \in L^1([0,\infty))$. Prove that the following function is differentiable at every $x > 0$:

$$g(x) = \int_0^{\infty}\frac{f(y)}{x+y}\,\mathrm{d}y.$$

   Solution. We directly verify differentiability. For all $x > 0$ and $0 < |h| < x/2$,

$$\begin{aligned}\left|\frac{g(x+h)-g(x)}{h} - \left(-\int_0^{\infty}\frac{f(y)}{(x+y)^2}\,\mathrm{d}y\right)\right| &= \left|\int_0^{\infty} f(y)\left(\frac{1}{h(x+y+h)} - \frac{1}{h(x+y)} + \frac{1}{(x+y)^2}\right)\mathrm{d}y\right| \\ &= |h|\cdot\left|\int_0^{\infty}\frac{f(y)}{(x+y)^2(x+y+h)}\,\mathrm{d}y\right| \\ &\le |h|\cdot\frac{2}{x^3}\int_0^{\infty}|f(y)|\,\mathrm{d}y,\end{aligned}$$

   which converges to 0 as $h \to 0$. The three integrals exist (and the linearity of integration applies) because the integrands are dominated by $|f(y)|/(hx)$, $|f(y)|/(hx)$, and $|f(y)|/x^2$, which are integrable with respect to y.

   Find an example of $f \in L^1([0,\infty))$ such that g is not differentiable at $x = 0$.

   Solution. Let $f(y) = 1/(y+1)^2$. The monotone convergence theorem implies that

$$\lim_{n\to\infty}\frac{g(\frac{1}{n})-g(0)}{\frac{1}{n}} = \lim_{n\to\infty} -\int_0^{\infty}\frac{f(y)}{y(y+\frac{1}{n})}\,\mathrm{d}y = -\int_0^{\infty}\frac{f(y)}{y^2}\,\mathrm{d}y = -\int_0^{\infty}\frac{2}{y} - \frac{1}{y^2} - \frac{2}{y+1} - \frac{1}{(y+1)^2}\,\mathrm{d}y = -\infty,$$

   which shows that g is not differentiable at 0.