Qualifying Exam: Real Analysis

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Spring 2023

1. Let $f\colon [0,1] \to \mathbb{R}$ be a Lebesgue-measurable function that is positive almost everywhere. Prove that if $E_1, E_2, \ldots$ are Lebesgue-measurable subsets of $[0,1]$ such that

$$\lim_{k\to\infty} \int_{E_k} f(x)\,\mathrm{d}x = 0,$$

then $\lim_{k\to\infty} m(E_k) = 0$.

Solution. Let $A_n = \{x : \frac{1}{n-1} > f(x) \geq \frac{1}{n}\}$, with $A_1 = \{x : f(x) \geq 1\}$. By Tonelli's theorem,

$$0 = \lim_{k\to\infty} \int f \cdot \mathbb{1}_{E_k}\,\mathrm{d}m \geq \lim_{k\to\infty} \int \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{1}_{A_n} \cdot \mathbb{1}_{E_k}\,\mathrm{d}m = \sum_{n=1}^{\infty} \frac{1}{n} \lim_{k\to\infty} \int \mathbb{1}_{A_n} \cdot \mathbb{1}_{E_k}\,\mathrm{d}m \geq 0.$$

A sum of nonnegative numbers is zero only if each of its terms is zero, so $\lim_{k\to\infty} m(A_n \cap E_k) = 0$ for all n. The equality $m(\bigsqcup_{n=1}^{\infty} A_n) = m(\{x : f(x) > 0\}) = 1$ and the dominated convergence theorem then imply that

$$\lim_{k\to\infty} m(E_k) = \lim_{k\to\infty} \sum_{n=1}^{\infty} m(A_n \cap E_k) = \sum_{n=1}^{\infty} \lim_{k\to\infty} m(A_n \cap E_k) = 0.$$

Remark. This is Problem 4 on the Fall 2021 exam.

2. Let $(g_n)_{n\geq 1}$ be a sequence of measurable functions on $[0,1]$ with the following properties:

   i. There exists $C < \infty$ such that $|g_n(x)| \leq C$ for every n and almost every $x \in [0,1]$;

   ii. For every $a \in [0,1]$, we have that $\lim_{n\to\infty} \int_0^a g_n(x)\,\mathrm{d}x = 0$.

   Prove that for every $f \in L^1([0,1])$,

$$(*) \qquad \lim_{n\to\infty} \int_0^1 f(x) g_n(x)\,\mathrm{d}x = 0.$$

Solution. Let $\mathscr{F}$ be the collection of functions in $L^1([0,1])$ that satisfy $(*)$, and let $\mathscr{A} = \{A : \mathbb{1}_A \in \mathscr{F}\}$. Property (ii) implies that $[0,a) \in \mathscr{A}$ for every $a \in [0,1]$. By the linearity of integration, $\mathscr{A}$ is an algebra; by the bounded convergence theorem, $\mathscr{A}$ is closed under monotone limits; by the monotone class lemma, $\mathscr{A}$ is a σ-algebra. Hence, $\mathscr{A}$ is the Borel σ-algebra, which means that $\mathscr{F}$ contains all measurable indicator functions on $[0,1]$. By the linearity of integration, $\mathscr{F}$ is closed under linear combinations. Because every $f \in L^1$ is the limit in L^1 of a sequence of simple functions $(\varphi_m)_m \subseteq \mathscr{F}$,

$$\begin{aligned}
\left|\lim_{n\to\infty}\int_0^1 f(x)g_n(x)\,\mathrm{d}x\right| &= \lim_{m\to\infty}\left|\lim_{n\to\infty}\int_0^1 \varphi_m(x)g_n(x)\,\mathrm{d}x - \lim_{n\to\infty}\int_0^1 f(x)g_n(x)\,\mathrm{d}x\right| \\
&\leq \lim_{m\to\infty}\lim_{n\to\infty}\int_0^1 |\varphi_m(x)-f(x)|\cdot|g_n(x)|\,\mathrm{d}x \\
&\leq \lim_{m\to\infty} C\int_0^1 |\varphi_m(x)-f(x)|\,\mathrm{d}x \\
&= 0.
\end{aligned}$$

Hence, $\mathscr{F} = L^1([0,1])$, as desired.

3. Let E be a measurable subset of $[0,1]$ with Lebesgue measure $m(E) = \frac{99}{100}$. Show that there exists $x \in [0,1]$ such that for every $r \in (0,1)$,

$$m(E \cap (x-r, x+r)) \geq \frac{r}{4}.$$

Hint: Use the Hardy–Littlewood inequality

$$m(\{x \in \mathbb{R} : \mathrm{M}f(x) \geq \alpha\}) \leq \frac{3}{\alpha}\|f\|_{L^1}.$$

Solution. What we want to show is the existence of $x \in [0,1]$ such that

$$\inf_{r>0} \frac{m(E \cap (x-r, x+r))}{2r} \geq \frac{1}{8}.$$

Let $f = \mathbb{1}_{[0,1]\setminus E}$. Equivalently, what we want to show is the existence of $x \in [0,1]$ such that

$$\mathrm{M}f(x) = \sup_{r>0} \frac{m((x-r, x+r) \setminus E)}{2r} \leq \frac{7}{8}.$$

By the Hardy–Littlewood inequality,

$$m\big(\{x \in [0,1] : \mathrm{M}f(x) \geq \tfrac{7}{8}\}\big) \leq \frac{3}{\left(\frac{7}{8}\right)} m([0,1] \setminus E) = \frac{24}{700} < 1.$$

Then, the set $\{x \in [0,1] : \mathrm{M}f(x) < \frac{7}{8}\}$ has positive Lebesgue measure and, in particular, is nonempty.

4. Let $f\colon [0,1] \to [0,1]$ be Lebesgue-measurable. Prove that for every $M > 0$, there exists $a \in [0,1]$ such that

$$\int_0^1 \frac{1}{|f(x) - a|}\,\mathrm{d}x \geq M.$$

Solution. By Tonelli's theorem, and because the codomain of f is $[0,1]$,

$$\int_0^1 \int_0^1 \frac{1}{|f(x) - a|}\,\mathrm{d}x\,\mathrm{d}a = \int_0^1 \int_0^1 \frac{1}{|f(x) - a|}\,\mathrm{d}a\,\mathrm{d}x = \infty.$$

Let $M > 0$. If $\int_0^1 1/|f(x) - a|\,\mathrm{d}x < M$ for every $a \in [0,1]$, then the integral above would be at most M, which is a contradiction; hence, there exists some $a \in [0,1]$ for which $\int_0^1 1/|f(x) - a|\,\mathrm{d}x \geq M$.

Remark. This problem is similar in spirit to Problem 3 on the Spring 2021 exam.