# Qualifying Exam: Real Analysis

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1. Prove that for almost every $x \in [0,1]$, there exist at most finitely many rational numbers $m/n$ such that

$$\left|x - \frac{m}{n}\right| < \frac{1}{(n\log n)^2}.$$

*Hint*: Consider intervals of length $2/(n\log n)^2$ centered at rational points $m/n$.

*Solution.* Following the hint, define

$$E_n = \bigcup_{m:\gcd(m,n)=1} \left(\frac{m}{n} - \frac{1}{(n\log n)^2}, \frac{m}{n} + \frac{1}{(n\log n)^2}\right).$$

Because $E_n$ is a disjoint union of at most $n$ intervals of length $2/(n\log n)^2$,

$$\sum_{n\geq 2} m(E_n) \leq \sum_{n\geq 2} \frac{2}{n(\log n)^2} < \infty.$$

Then, the Borel–Cantelli lemma implies that

$$m(\{x \in [0,1] : \text{there exist at most finitely many } n \text{ such that } x \in E_n\}) = m\left([0,1] \setminus \limsup_{n\to\infty} E_n\right) = 1.$$

2. Let $S$ be a closed subset of $\mathbb{R}$, and let $f \in L^1([0,1])$. Suppose that for every measurable subset $E$ of $[0,1]$ with $m(E) > 0$,

$$\frac{1}{m(E)} \int_E f(x)\,\mathrm{d}x \in S.$$

Prove that $f(x) \in S$ for almost every $x \in [0,1]$.

*Solution.* Let $U = \mathbb{R} \setminus S$, and assume for the sake of contradiction that $m(f^{-1}(U)) > 0$. The nonempty open set $U$ is a countable union of open intervals, one of which must satisfy $m(f^{-1}((a,b))) > 0$. Continuity from below then yields $n$ such that $m(f^{-1}([a+\frac{1}{n}, b-\frac{1}{n}])) > 0$. For the measurable set $E = f^{-1}([a+\frac{1}{n}, b-\frac{1}{n}])$,

$$a + \frac{1}{n} \leq \frac{1}{m(E)} \int_E f(x)\,\mathrm{d}x \leq b - \frac{1}{n},$$

which contradicts the disjointness of $[a+\frac{1}{n}, b-\frac{1}{n}]$ and $S$.

3. Compute

$$\lim_{n\to\infty} \int_0^1 \frac{1+nx}{(1+x)^n}\,\mathrm{d}x.$$

*Solution.* For all $x > 0$ and for every $n \geq 2$,

$$\frac{1+nx}{(1+x)^n} \leq \frac{1+nx}{\binom{n}{2}x^2} = \frac{2}{x^2} \cdot \frac{1}{n(n-1)} + \frac{2}{x} \cdot \frac{1}{n-1}.$$

The binomial theorem implies that $(1+nx)/(1+x)^n \leq 1$, so the bounded convergence theorem implies that

$$\lim_{n\to\infty} \int_0^1 \frac{1+nx}{(1+x)^n}\,\mathrm{d}x = \int_0^1 \lim_{n\to\infty} \frac{1+nx}{(1+x)^n}\,\mathrm{d}x = \int_0^1 0\,\mathrm{d}x = 0.$$

4. Let $(X, \mathscr{A}, \mu)$ be a finite measure space, and let $(f_n)_{n\geq 1}$ be a sequence of nonnegative measurable functions on $X$. Prove that $f_n \to 0$ in measure if and only if

$$\lim_{n\to\infty} \int_X \frac{f_n(2+f_n)}{(1+f_n)^2}\,\mathrm{d}\mu = 0.$$

*Solution.* Let $\varphi(y) = \frac{y(2+y)}{(1+y)^2} = 1 - \frac{1}{(1+y)^2}$. The key observation is that for all $y \geq 0$,

$$3\min\{1, y\} \geq \varphi(y) \geq \tfrac{1}{2}\min\{1, y\},$$

a consequence of which is the logical equivalence of $\int \varphi(f_n)\,\mathrm{d}\mu \to 0$ and $\int \min\{1, f_n\}\,\mathrm{d}\mu \to 0$.

Let $g_n = \min\{1, f_n\}$. If $f_n \to 0$ in measure, then $g_n \to 0$ in measure because $\mu(\{x : f_n(x) \geq \varepsilon\}) \geq \mu(\{x : g_n(x) \geq \varepsilon\})$, and the bounded convergence theorem implies that $\int g_n\,\mathrm{d}\mu \to 0$. If $\int g_n\,\mathrm{d}\mu \to 0$, then $g_n \to 0$ in measure, and $f_n \to 0$ in measure because $\mu(\{x : g_n(x) \geq \varepsilon\}) = \mu(\{x : f_n(x) \geq \varepsilon\})$ for $0 < \varepsilon \leq 1$.