Qualifying Exam: Real Analysis

Unofficial solutions by Alex Fu

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1. Suppose that f is a bounded nonnegative function on a measure space $(X, \mathscr{A}, \mu)$ with $\mu(X) = \infty$. Prove that f is integrable if and only if

$$(*) \qquad \sum_{n=1}^{\infty} \frac{1}{2^n} \mu(\{x \in X : f(x) > 2^{-n}\}) < \infty.$$

Solution. Let $E_t = \{x \in X : f(x) > t\}$. Note that $t \mapsto \mu(E_t)$ is nonincreasing. If f is integrable, then the tail-sum formula and the monotone convergence theorem imply that

$$\infty > \int_X f \, \mathrm{d}\mu = \int_0^\infty \mu(E_t) \, \mathrm{d}t \geq \sum_{n=0}^{\infty} \int_{2^{-n}}^{2^{-n+1}} \mu(E_t) \, \mathrm{d}t \geq \sum_{n=0}^{\infty} 2^{-n} \cdot \mu(E_{2^{-n+1}}).$$

Conversely, if $(*)$ holds, then

$$\infty > \sum_{n=1}^{\infty} 2^{-n} \cdot \mu(E_{2^{-n}}) \geq \sum_{n=1}^{\infty} \int_{2^{-n}}^{2^{-n+1}} \mu(E_t) \, \mathrm{d}t = \int_0^1 \mu(E_t) \, \mathrm{d}t,$$

and $\mu(E_1) \leq \mu(E_{1/2}) < \infty$. Hence, f is integrable:

$$\int_X f \, \mathrm{d}\mu = \int_0^\infty \mu(E_t) \, \mathrm{d}t \leq \int_0^1 \mu(E_t) \, \mathrm{d}t + \int_1^{\|f\|_{L^\infty}} \mu(E_1) \, \mathrm{d}t < \infty.$$

2. Prove that for any $f : [0,1] \to \mathbb{R}$, the set of points where f is continuous is a Lebesgue-measurable set.

Solution. Let $E_{\varepsilon,\delta} = \{x \in [0,1] : |f(y) - f(z)| < \varepsilon \text{ for all } y, z \in \mathrm{Ball}(x, \delta)\}$, so that the continuity set of f is

$$\bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} E_{1/n, 1/m}.$$

Let $U_n = \bigcup_{m=1}^{\infty} E_{1/n,1/m}$. If $x \in U_n$, there exists m such that $x \in E_{1/n,1/m}$. Then, for every $y \in \mathrm{Ball}(x, \frac{1}{2m})$, it holds that $\mathrm{Ball}(y, \frac{1}{2m}) \subseteq \mathrm{Ball}(x, \frac{1}{m})$ and $y \in E_{1/n, 1/(2m)} \subseteq U_n$. In other words, $\mathrm{Ball}(x, \frac{1}{2m}) \subseteq U_n$, which shows that U_n is open. It follows that the continuity set of f is Lebesgue-measurable.

3. Let f be a Lebesgue-integrable function on $\mathbb{R}$, and let $\beta \in (0,1)$. Prove that for almost every $\alpha \in \mathbb{R}$,

$$\int_0^\infty \frac{|f(x)|}{|x-\alpha|^\beta}\,\mathrm{d}x < \infty.$$

Solution. Let $n \geq 1$. The continuous function of x

$$I_n(x) := \int_{-n}^{n} \frac{1}{|x-\alpha|^\beta}\,\mathrm{d}\alpha = \frac{x^{1-\beta}(|1+\frac{n}{x}|^{1-\beta} - |1-\frac{n}{x}|^{1-\beta})}{1-\beta}$$

is asymptotic to $x^{1-\beta}(1+(1-\beta)\frac{n}{x} - (1-(1-\beta)\frac{n}{x})) \propto nx^{-\beta}$ as $x \to \infty$, and hence satisfies $\lim_{x\to\infty} I_n(x) = 0$. It follows that I_n is essentially bounded. Then, Tonelli's theorem justifies

$$\int_{-n}^{n}\int_0^\infty \frac{|f(x)|}{|x-\alpha|^\beta}\,\mathrm{d}x\,\mathrm{d}\alpha = \int_0^\infty\int_{-n}^{n} \frac{|f(x)|}{|x-\alpha|^\beta}\,\mathrm{d}\alpha\,\mathrm{d}x = \int_0^\infty |f(x)|\cdot I_n(x)\,\mathrm{d}x \leq \|f\|_{L^1}\cdot\|I_n\|_{L^\infty} < \infty,$$

which implies that $\int_0^\infty |f(x)|/|x-\alpha|^\beta\,\mathrm{d}x < \infty$ for almost every $\alpha \in [-n,n]$.

Remark. This problem is similar in spirit to Problem 4 on the Spring 2023 exam.

4. Let $(f_n)_{n\geq 1}$ be a sequence of real-valued functions on $[a,b]$ that converges pointwise, let $f = \lim_{n\to\infty} f_n$, and let $V_a^b(f)$ be the total variation of f on $[a,b]$. Show that

$$V_a^b(f) \leq \liminf_{n\to\infty} V_a^b(f_n).$$

Solution. Recall that

$$V_a^b(f) = \sup\left\{\sum_{i=1}^m |f(x_i) - f(x_{i-1})| : m \geq 1, a = x_0 < \cdots < x_m = b\right\}.$$

Let $m \geq 1$, and let $a = x_0 < \cdots < x_m = b$. Then,

$$\sum_{i=1}^m |f(x_i) - f(x_{i-1})| = \liminf_{n\to\infty}\sum_{i=1}^m |f_n(x_i) - f_n(x_{i-1})| \leq \liminf_{n\to\infty} V_a^b(f_n).$$

Taking the supremum of both sides over all $m \geq 1$ and all $a = x_0 < \cdots < x_n = b$ yields

$$V_a^b(f) \leq \liminf_{n\to\infty} V_a^b(f_n).$$