20sp. 1

$f : \mathbb{R} \to \mathbb{R}$ strictly incr., cont.

T/F,

If $A \subset \mathbb{R}$ Lebesgue measurable then $f^{-1}(A)$ Lebesgue measurable.

False. Let $C \subset [0,1]$ be the cantor set. Let $g(x)$ be the cantor function. Define $h(x) = g(x) + x$, $h : [0,1] \to [0,2]$.
g, x continuous $\Rightarrow$ h continuous. $h'(x) = 1$ a.e. since $g'(x) = 0$ a.e.,
$\Rightarrow$ h strictly increasing. Thus $h^{-1} : [0,2] \to [0,1]$ well defined, continuous, and strictly increasing.

Consider
$$\begin{aligned}
m(h([0,1]\setminus C)) &= m(h(\cup_1^\infty (a_i, b_i))), \quad \text{[where } \{(a_i,b_i)\} \text{ are the intervals taken out to form } C. \\
&= m(\cup_1^\infty h((a_i,b_i))) \\
&= \textstyle\sum_1^\infty m(h((a_i,b_i))), \quad \text{since } h \text{ strictly increasing} \\
&= \textstyle\sum_1^\infty m((h(a_i), h(b_i))) \\
&= \textstyle\sum_1^\infty m((g(a_i)+a_i, g(b_i)+b_i)) \\
&= \textstyle\sum_1^\infty b_i - a_i, \quad \text{since } g(a_i) = g(b_i) \\
&= m([0,1]\setminus C) \\
&= 1
\end{aligned}$$

Thus, $m(h(C)) = m([0,2]) - m(h([0,1]\setminus C)) = 2 - 1 = 1$.

Then, $\exists$ non-measurable set $E \subset h(C)$, since $m(h(C)) > 0$.

But $h^{-1}(E) \subset h^{-1}(h(C)) = C$, a null set in Lebesgue measure.
Since Lebesgue measure complete, $h^{-1}(E)$ Lebesgue measurable.

So take $f := h^{-1}$, with tails that go to $\pm\infty$ on $\mathbb{R}$, and take $A := h^{-1}(E)$, Lebesgue measurable.

Then $f^{-1}(A) = h(h^{-1}(E)) = E$, not Lebesgue measurable.

Note, $h^{-1}(E)$ special because it's Lebesgue measurable, but not Borel.
If A is just Borel measurable, this is true.

20 sp. 2

? How

$$\lim_{n\to\infty} n \int_{1/n}^{1} \frac{\cos(x+\frac{1}{n}) - \cos x}{x}\, dx$$

$$= \lim_{h\to 0} \frac{1}{h}\int_h^1 \frac{\cos(x+h) - \cos x}{x}\, dx$$

$$= \int_0^1 \frac{1}{x} \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} \chi_{[h,1]}\, dx \quad * \text{ by DCT}$$

$$= \int_0^1 \frac{\frac{d}{dx}\cos(x)}{x}\, dx$$

$$= \int_0^1 \frac{-\sin(x)}{x}\, dx < \infty, \quad \text{since } \frac{-\sin(x)}{x} \text{ continuous on } (0,1]$$

and $\lim_{x\to 0} \frac{-\sin(x)}{x} = \lim_{x\to 0} \frac{-\cos(x)}{1} = -1$.

* Let $f_h = \frac{1}{x} \frac{\cos(x+h) - \cos x}{h} \chi_{[h,1]}$

By MVT, $\left| \frac{\cos(x+h) - \cos x}{h} \right| \leq |\cos'(x+h)|$, for $x \in (0,1]$, h small.

$= |-\sin(x+h)|$

Thus, $|f_h| \leq \frac{\sin(x+h)}{x} \chi_{[h,1]}$ $\forall\, x \in (0,1]$, $\forall h$.

$\lim_{h\to 0} f_h(h) = \lim_{h\to 0} \frac{\sin(2h)}{h} = \lim_{h\to 0} 2\sin(h) = 2.$

$\frac{d}{dh} f_h(h) = \frac{d}{dh} \frac{\sin(2h)}{h} = \frac{2h\cos(2h) - \sin(2h)}{h^2} < 0$ when $h > 0$, because

$$2h\cos(2h) - \sin(2h) < 0 \iff 2h\cos(2h) < \sin(2h)$$
$$\iff \tan(2h) > 2h$$
$$\Leftarrow 2h \in (0, \pi/2).$$

So, $f_h(h) \nearrow 2$ as $h \searrow 0$.

Similarly $\frac{d}{dx} \frac{\sin(x+h)}{x} = \frac{x\cos(x+h) - \sin(x+h)}{x^2} < 0$ for $x \in (0, \pi/2)$.

Hence $|f_h| \leq 2 \quad \forall\ x \in (0,1],\ \forall h$.

which is clearly integrable on $[0,1]$.

20 sp. 3

$f: \mathbb{R} \to \mathbb{R}$, $M > 0$. Prove (a) $\Longleftrightarrow$ (b) where
(a) $|f(x) - f(y)| \leq M|x-y| \quad \forall x, y \in \mathbb{R}$.
(b) f abs. cont. with $|f'(x)| \leq M \quad \forall x \in \mathbb{R}$.

(a) $\Rightarrow$ (b): NTS $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. for any finite collection of disjoint intervals, $\{(a_i, b_i)\}$, we have

$$\sum (b_i - a_i) < \delta = \sum |f(b_i) - f(a_i)| < \varepsilon.$$

Take $\delta = \varepsilon / M$, then
$$\sum |f(b_i) - f(a_i)| \leq \sum M|b_i - a_i| = M \sum (b_i - a_i) < M\delta = \varepsilon$$

And $|f'(x)| = \left| \lim_{y \to x} \frac{f(y) - f(x)}{y - x} \right| = \lim_{y \to x} \frac{|f(y) - f(x)|}{|y - x|} \leq M$
since $\frac{|f(y) - f(x)|}{|y - x|} \leq M \quad \forall x, y \in \mathbb{R}$.

(b) $\Rightarrow$ (a): f abs cont $\Rightarrow$ f abs. cont. on $[-k, k]$ for any $k \in \mathbb{Z}$.

$\Rightarrow$ f diff a.e. on $[-k, k]$ and
$\int_a^b f'(x)\,dx = f(b) - f(a) \quad \forall -k \leq a \leq b \leq k$.

$|f'(x)| \leq M \; \forall x \in \mathbb{R} \Rightarrow \int_a^b f'(x)\,dx \leq M(b-a)$
$\Rightarrow f(b) - f(a) \leq M(b-a)$

and $\int_a^b f'(x)\,dx \geq -M(b-a)$
$\Rightarrow f(b) - f(a) \geq -M(b-a)$

Which is to say $|f(b) - f(a)| \leq M|b-a| \quad \forall a, b \in [-k, k]$.

Since k arbitrary, for any $x, y \in \mathbb{R}$, the result holds, since $\exists k \in \mathbb{Z}$, s.t. $x, y \in [-k, k]$.

<u>20 sp. 4</u>

m Lebesgue measure, m^* Lebesgue outer measure.

$$m_*(A) = \sup\{m(K) : K \subset A,\ K \text{ compact}\}.$$

Prove $m^*(A) = m_*(A)$ and $m^*(A) < \infty \Rightarrow A$ Lebesgue measurable.

$$m^*(A) = \inf\{\sum b_i - a_i : A \subset \bigcup_i^\infty (a_i, b_i)\}$$
$$= \sup\{m(K) : K \subset A,\ K \text{ compact}\} < \infty$$

Claim: A is m^*-measurable $\Rightarrow$ A is Lebesgue measurable.

Claim: $\forall E \subset \mathbb{R}$, $m^*(E) = m^*(E \cap A) + m^*(E \cap A^c)$.

Know $m^*(E) \le m^*(E \cap A) + m^*(E \cap A^c)$ since m^* outer measure.

$\exists$ K compact, $\{(a_i, b_i)\}$ s.t. $K \subset A \subset \bigcup_i^\infty (a_i, b_i)$
and $\sum (b_i - a_i) - m(K) < \varepsilon$.

Let $E \subset \mathbb{R}$. $m^*(E) = \inf\{\sum b_i - a_i : E \subset \bigcup_i^\infty (a_i, b_i)\}$

$m^*(E \cap A) = \inf\{\sum b_i - a_i : E \cap A \subset \bigcup_i^\infty (a_i, b_i)\}$

$m^*(E \cap A^c) = \inf\{\sum b_i - a_i : E \cap A^c \subset \bigcup_i^\infty (a_i, b_i)\}$

$\exists$ K compact s.t. $m(K) > m^*(A) - \varepsilon$.

K compact $\Rightarrow K \in \mathcal{B}_{\mathbb{R}} \Rightarrow K$ m^*-measurable $\Rightarrow m^*(E) = m^*(K \cap E) + m^*(K^c \cap E)$.

$$E \cap A = (E \cap K) \cup (E \cap (A \setminus K))$$

$$\begin{aligned}
m^*(E \cap A) &\le m^*(E \cap K) + m^*(E \cap (A \setminus K)) \\
&= m^*(E) - m^*(K^c \cap E) + m^*(E \cap (A \setminus K)) \\
&\le m^*(E) - m^*(K^c \cap E) + m^*(A \setminus K) \\
&< m^*(E) - m^*(K^c \cap E) + \varepsilon \\
&\le m^*(E) - m^*(A^c \cap E) + \varepsilon
\end{aligned}$$

$\Rightarrow m^*(E \cap A) + m^*(A^c \cap E) \le m^*(E) + \varepsilon$

$\Rightarrow m^*(E \cap A) + m^*(A^c \cap E) \le m^*(E)$

Done.

$F(x) = x.$

$\mathcal{A} = \{$ finite disjoint unions of h-intervals $\}$, an algebra.

$\mu_0\left(\bigcup_1^n (a_j, b_j]\right) = \sum_1^n F(b_j) - F(a_j) = \sum_1^n b_j - a_j$, a premeasure on $\mathcal{A}$.

$\mathcal{B}_{\mathbb{R}}$ = Borel σ-algebra $= \sigma(\mathcal{A})$.

$\mu^*(E) = \inf\{\sum_1^\infty \mu_0(A_j) : A_j \in \mathcal{A},\ E \subset \bigcup_1^\infty A_j\}$, an outer measure.
$= \inf\{\sum_1^\infty b_j - a_j : E \subset \bigcup_1^\infty (a_j, b_j)\}.$

$\mathcal{M} = \{\mu^*$-measurable sets$\}$

Cartheodory $\Rightarrow$ $\mathcal{M}$ σ-algebra, $\mu^*|_{\mathcal{M}}$ complete measure.

$1.13 \Rightarrow \mathcal{A} \subset \mathcal{M} \Rightarrow \mathcal{B}_{\mathbb{R}} \subset \mathcal{M}$

Define Lebesgue measure $m = \mu^*|_{\mathcal{M}}$