19 Sp. 1

1. Find the limit
$$\lim_{j\to\infty}\int_{-1}^{1}\frac{1-e^{-\frac{x^2}{j}}}{x^2}dx.$$

$$\lim_{j\to\infty}\int_{-1}^{1}\frac{1-e^{-x^2/j}}{x^2}dx = \int_{-1}^{1}\lim_{j\to\infty}\frac{1-e^{-x^2/j}}{x^2}dx \quad \text{by DCT}$$
$$= \int_{-1}^{1} 0\, dx = 0.$$

DCT:

$$\frac{d}{dj}\frac{1-e^{-x^2/j}}{x^2} = \frac{-e^{-x^2/j}(x^2/j^2)}{x^2} \leq 0 \quad \forall x \neq 0,$$
$$\Rightarrow \frac{1-e^{-x^2/j}}{x^2} \leq \frac{1-e^{-x^2}}{x^2} \quad \forall x\neq 0, \forall j \geq 1.$$

and $\frac{1-e^{-x^2}}{x^2}$ is continuous on $[-1,0)$ and $(0,1]$, with $\lim_{x\to 0}(1-e^{-x^2})/x^2 = 1$. Thus $\exists$ M s.t. $(1-e^{-x^2})/x^2 \leq M \quad \forall x \in [-1,1]\setminus\{0\}$,

$$\Rightarrow \int_{-1}^{1}\frac{1-e^{-x^2/j}}{x^2}dx \leq \int_{-1}^{1} M < \infty$$

Hence, DCT may be used.

Fix x. $\quad \lim_{j\to\infty}\frac{1-e^{-x^2/j}}{x^2} = 0$

Fix j. $\quad \lim_{x\to 0}\frac{1-e^{-x^2/j}}{x^2} = \frac{0}{0}$

$$LH \Rightarrow \lim_{x\to 0}\frac{(2x/j)e^{-x^2/j}}{2x} = \frac{0}{0}$$

$$LH \Rightarrow \lim_{x\to 0}\frac{\frac{2}{j}e^{-x^2/j} + (2x/j)\cancel{(-2x/j)}e^{-x^2/j}}{2} = \frac{\frac{2}{j}}{2} = \frac{1}{j}$$

19sp.2

2. Let F be absolutely continuous and $F, F' \in L^1(\mathbb{R}, m)$, where $m(dx) = dx$ is the Lebesgue measure. Prove that

$$\int_{-\infty}^{\infty} F'(x)\,dx = 0.$$

For $M \in \mathbb{N}$, on $[-M, M]$, we have

$$\int_{-M}^{M} F'(x)\,dx = F(M) - F(-M).$$

Assume $\int_{-\infty}^{\infty} F'(x)\,dx = C > 0$. ($>0$ WLOG).

Since $F' \in L^1$, $\exists\ M \in \mathbb{N}$ s.t. $\forall\ m \geq M$,
$\int_{\mathbb{R} \setminus [-m, m]} F'(x)\,dx < C/2$.

Then $\forall\ m \geq M$,

$$\int_{-\infty}^{\infty} F'(x)\,dx = \int_{[-m,m]} F'(x)\,dx + \int_{\mathbb{R} \setminus [-m,m]} F'(x)\,dx = C$$

$$\Rightarrow \int_{[-m,m]} F'(x)\,dx = C - \int_{\mathbb{R} \setminus [-m,m]} F'(x)\,dx > C/2$$

$$\Rightarrow F(m) - F(-m) > C/2 \qquad \forall\ m \geq M.$$

Then $\int_{m \geq M} F(m) - F(-m)\,dm(m) = \infty$

But $\int_{m \geq M} F(m) - F(-m)\,dm(m) \leq \int_{m \geq M} |F(m)| + |F(-m)|$
$\leq 2\int_{\mathbb{R}} |F| < \infty \quad \Rightarrow\Leftarrow.$

19 sp. 3

3. Let $(X, \mathcal{M}, \mu)$ be a measure space. Assume that $h \circ f$ is integrable for every continuous function $h : \mathbb{R} \to \mathbb{R}$. Prove that there is $a > 0$ so that

$$\mu(\{x : |f(x)| > a\}) = 0.$$

Let $h(x) = h(-x) > 0 \ \forall x$. Let $h(n) \geq n / \mu(\{x : |f(x)| > n\}) \ \forall n \in \mathbb{N}$, with $\mu(\{x : |f(x)| > n\}) > 0$, and make sure h is continuous and increasing as $|x|$ increases.

Then
$$\begin{aligned} \int_X |h \circ f| \, d\mu &\geq \int_X |h \circ f| \, \chi_{\{x : |f(x)| > n\}} \, d\mu \\ &\geq \int_X h(n) \, \chi_{\{x : |f(x)| > n\}} \, d\mu \\ &= h(n) \, \mu(\{x : |f(x)| > n\}) \\ &\geq n \end{aligned}$$

$\forall n \in \mathbb{N}$ s.t. $\mu(\{x : |f(x)| > n\}) > 0$.

Thus, since $\int_X |h \circ f| \, d\mu < \infty$, there must be some $a > 0$ s.t. $\mu(\{x : |f(x)| > a\}) = 0$.

19 sp. 4

4. (i) Show that for every $\varepsilon > 0$ there exists a non-negative $f \in L^1([0,1], m)$ such that $f(x) = 0$ on the set of measure $\geq 1 - \varepsilon$ and

$$\int_a^b f(x)\,dx > 0$$

for all $0 < a < b < 1$.

(ii) Show that for each $\varepsilon > 0$ there exists an absolutely continuous strictly increasing h on $[0,1]$ so that $h'(x) = 0$ on a set of measure $\geq 1 - \varepsilon$.

(i) Enumerate the rationals: $r_1, r_2, \ldots$

Let $f(x) = \sum_{n=1}^{\infty} \chi_{[r_n, r_n + \varepsilon/2^{n+1}] \cap [0,1]}(x)$

Then $f(x) = 1$ on a set of measure $\leq \sum_{n=1}^{\infty} \varepsilon/2^{n+1} < \varepsilon$.
$\Rightarrow$ $f(x) = 0$ on a set of measure $\geq 1 - \varepsilon$.

Let $0 < a < b < 1$. $\exists$ rational r_n s.t. $a < r_n < b$.

Then $\int_a^b f(x)\,dx \geq \int_a^b \chi_{[r_n, r_n + \varepsilon/2^{n+1}]}(x)\,dx$
$\geq \min\{b - r_n, \varepsilon/2^{n+1}\}$
> 0.

(ii) Let f satisfy (i). Define $h(x) = \int_0^x f(y)\,dy$. for $x \in [0,1]$

Then h is absolutely continuous on $[0,1]$, and
$f(x) = h'(x)$ a.e. $\Rightarrow$ $h'(x) = 0$ on a set of measure $\geq 1 - \varepsilon$.
And $\forall$ $0 < a < b < 1$, we have $h(b) - h(a) = \int_a^b f(x)\,dx > 0$
$\Rightarrow$ h is strictly increasing.