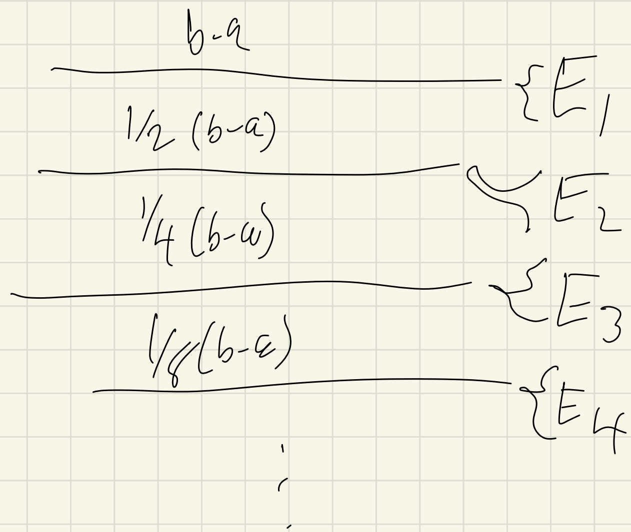
18 sp. 1

Problem 1. Let $-\infty < a < b < \infty$ and suppose $\mathcal{B}$ is a countable collection of closed subintervals of (a, b). Give the proof that there is a countable pairwise-disjoint subcollection $\mathcal{B}' \subset \mathcal{B}$ such that

$$\bigcup_{I \in \mathcal{B}} I \subset \bigcup_{I \in \mathcal{B}'} \tilde{I},$$

where $\tilde{I}$ denotes the 5-times enlargement of I; thus if $I = [x - \rho, x + \rho]$ then $\tilde{I} = [x - 5\rho, x + 5\rho]$.

Let $E_n := \{I$



18 sp. 2

Problem 2. Assume that f is absolutely continuous on $[0,1]$, and assume that $f' = g$ a.e., where g is a continuous function. Prove that f is continuously differentiable on $[0,1]$.

Let $\varepsilon > 0$, $x \in [0,1]$. g continuous $\Rightarrow \exists\, \delta > 0$ s.t.
$\forall\, y \in (x-\delta, x+\delta) \cap [0,1]$, $|g(y) - g(x)| < \varepsilon$.

f absolutely continuous and $g = f'$ a.e. $\Rightarrow$ on any closed interval $[a,b] \subset [0,1]$, we have $\int_a^b g\, dm = f(b) - f(a)$.

We have $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} g\, dm$

Thus as the sequence $h_n \longrightarrow 0$, $\exists\, N \in \mathbb{N}$ s.t.
$\forall\, n \geq N$, $|h_n| < \delta \Rightarrow |g(x+h_n) - g(x)| < \varepsilon \quad \forall\, n \geq N$.
$\Rightarrow (g(x)-\varepsilon)h = \int_x^{x+h} g(x) - \varepsilon\, dm \leq \int_x^{x+h} g\, dm \leq \int_x^{x+h} g(x) + \varepsilon\, dm = (g(x)+\varepsilon)h$

$$\Rightarrow \quad g(x) - \varepsilon \leq f'(x) \leq g(x) + \varepsilon$$

$\Rightarrow \quad f'(x) = g(x)$

$\Rightarrow \quad f' = g \quad \forall\, x \quad \Rightarrow \quad f'$ continuous $\Rightarrow$ f cont. diff.

18 sp. 3

Problem 3. Let $(X, \mathcal{M}, \mu)$ be a measure space such that $\mu(X) = 1$. Let $A_1, A_2, ..., A_{50} \in \mathcal{M}$. Assume that almost every point in X belongs to at least 10 of these sets. Prove that at least one of the sets has measure greater than or equal to 1/5.

Consider $\int_X \sum_1^{50} \chi_{A_i}\, d\mu = \sum_1^{50} \mu(A_i)$ since $\sum_1^{50} \chi_{A_i} \in L^+$

$\sum_1^{50} \chi_{A_i}(x) \geq 10$ a.e. $\Rightarrow \int_X \sum_1^{50} \chi_{A_i}\, d\mu \geq \int_X 10\, d\mu = 10$.

Thus, $\sum_1^{50} \mu(A_i) \geq 10$

If $\mu(A_i) < \frac{1}{5}$ $\forall i$, then $\sum_1^{50} \mu(A_i) < \sum_1^{50} \frac{1}{5} = 10 \Rightarrow\Leftarrow$

Hence $\mu(A_i) \geq \frac{1}{5}$ for some i.

18 sp. 4

Problem 4. Let $f : [0, \infty) \to \mathbb{R}$ be absolutely continuous on every closed subinterval of $[0, \infty)$ and

$$f(x) = f(0) - \int_0^x g(t)dt, \qquad \text{for } x \geq 0,$$

where $g \in \mathcal{L}^1([0, \infty))$. Show that

$$\int_0^\infty \frac{f(2x) - f(x)}{x} dx = (\log 2) \int_0^\infty g(t)dt.$$

Sign?

$$\int_0^\infty \frac{f(2x)-f(x)}{x}\,dx = \int_0^\infty \frac{f(0) - \int_0^{2x} g(t)\,dt - f(0) + \int_0^x g(t)\,dt}{x}\,dx$$

$$= \int_0^\infty \frac{-\int_x^{2x} g(t)\,dt}{x}\,dx$$

$$= \int_0^\infty \int_0^\infty -\frac{1}{x} g(t) \chi_{[x,2x]}(t)\,dt\,dx$$

$$= \int_0^\infty \int_0^\infty -\frac{1}{x} g(t) \chi_{[t/2,t]}(x)\,dt\,dx$$

$$= \int_0^\infty -g(t) \int_0^\infty \frac{1}{x} \chi_{[t/2,t]}(x)\,dx\,dt \quad *$$

$$= \int_0^\infty -g(t)\left[\ln(t) - \ln(t/2)\right]dt$$

$$= \ln(2) \int_0^\infty g(t)\,dt.$$

$*$ Tonell $\Rightarrow$ $\int \left| -\frac{1}{x} g(t) \chi_{[t/2,t]}(x) \right| (dm(t) \times dm(x))$

$$= \int_0^\infty \int_0^\infty \left| -\frac{1}{x} g(t) \chi_{[t/2,t]}(x) \right| dx\,dt$$

$$= \int_0^\infty |-g(t)| \int_0^\infty \left| \frac{1}{x} \chi_{[t/2,t]}(x) \right| dx\,dt$$

$$= \ln(2) \int_0^\infty |g(t)| < \infty$$

$\Rightarrow$ $-\frac{1}{x} g(t) \chi_{[t/2,t]}(x) \in L^1(m(t) \times m(x))$

$\Rightarrow$ Fubini may be used.