17 sp. 1

Problem 1. Assume that f is a positive absolutely continuous function on $[0, 1]$. Prove that $1/f$ is also absolutely continuous on $[0, 1]$.

f continuous on $[0,1]$, compact $\Rightarrow$ f has a minimum.
f positive $\Rightarrow$ $\min\{f(x)\} > 0$. Call this ℓ.

Let $\varepsilon > 0$. f absolutely continuous $\Rightarrow$ $\exists\ \delta > 0$ s.t. for any collection of intervals $\{(a_i, b_i)\}$, we have

$$\Sigma (b_i - a_i) < \delta \implies \Sigma |f(b_i) - f(a_i)| < \varepsilon \ell^2$$

Thus for any collection of intervals with $\Sigma(b_i - a_i) < \delta$, we also have

$$\begin{aligned}\sum \left| \frac{1}{f(b_i)} - \frac{1}{f(a_i)} \right| &= \sum \frac{|f(a_i) - f(b_i)|}{f(a_i) f(b_i)} \\ &\leq \sum \frac{|f(a_i) - f(b_i)|}{\ell^2} \\ &= 1/\ell^2\ \Sigma |f(a_i) - f(b_i)| \\ &< \varepsilon.\end{aligned}$$

Hence, $1/f$ is absolutely continuous.

$\square$

17 sp. 2

Problem 2. Assume that E is Lebesgue measurable.

(a) Suppose $m(E) < \infty$, where m is the Lebesgue measure. Show that

$$f(x) = \int \chi_E(y)\chi_E(y-x)dm(y)$$

is continuous. (Here, χ_A denotes the characteristic function of a set $A \subset \mathbb{R}$).

(b) Suppose $0 < m(E) \leq \infty$. Show that $S = E - E = \{x - y : x, y \in E\}$ contains an open interval $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$.

?

Looks like Kayla's works

a) Note $f(x) = \int \chi_E(y)\chi_E(y-x)\,dm(y) = \int \chi_E(y)\chi_{E+x}(y)\,dm(y) = m(E \cap E+x)$.

First, assume $E = (a,b)$ is an open interval.
Then $f(x) = m((a,b) \cap (a+x, b+x)) = \chi_{0<x\leq b-a}(x)(b-a-x) + \chi_{a-b\leq x\leq 0}(x)(b-a+x)$
which is continuous.

Try again.

By linearity, $f(x)$ is continuous when E is a ~~countable~~ finite union of disjoint open intervals, with $m(E) < \infty$.

Now, consider arbitrary measurable E with $m(E) < \infty$.
We can find a countable union of disjoint open intervals s.t.
$U = \bigcup (a_i, b_i) \supset E$ and $m(U \setminus E) < \varepsilon$, for any $\varepsilon > 0$.
For clarity denote $f_E(x) := m(E \cap E+x)$ and $f_U(x) = m(U \cap U+x)$.

$$\begin{aligned} f_U(x) - f_E(x) = m(U \cap U+x) - m(E \cap E+x) &= m((U \cap U+x) \setminus (E \cap E+x)) \\ &\leq m(U \setminus E) + m(U+x \setminus E+x) \\ &< 2\varepsilon. \qquad \forall x. \end{aligned}$$

Thus, for $z \to x$ we can make $f_E(z) \to f_E(x)$,
since ε is arbitrary, and f_U is continuous $\forall\, U$ constructed as above.

b) $S = E - E = \bigcup_{y \in E} E - y$.

Try Kayla's.

17 sp. 3

Problem 3. Assume that f is a continuous function on $[0,1]$. Prove that

$$\lim_{n\to\infty}\int_0^1 nx^{n-1}f(x)dx = f(1).$$

Let $\varepsilon > 0$. f continuous $\Rightarrow$ $\exists\ \delta > 0$ s.t. $\forall\ x \in (1-\delta, 1]$, we have $|f(x) - f(1)| < \varepsilon$.

Then $\lim\limits_{n\to\infty} \int_0^1 nx^{n-1}f(x)dx = \lim\limits_{n\to\infty} \int_0^{1-\delta} nx^{n-1}f(x)dx + \lim\limits_{n\to\infty} \int_{1-\delta}^1 nx^{n-1}f(x)dx$

First, assume $f(1) = 0$.

Then
$$\lim_{n\to\infty} \left| \int_{1-\delta}^1 nx^{n-1}f(x)dx \right| \le \lim_{n\to\infty} \int_{1-\delta}^1 nx^{n-1}|f(x)|dx$$
$$\le \varepsilon \lim_{n\to\infty} \int_{1-\delta}^1 nx^{n-1}dx$$
$$= \varepsilon \lim_{n\to\infty} x^n\Big]_{1-\delta}^1$$
$$= \varepsilon.$$

Thus. $\int_{1-\delta}^1 nx^{n-1}f(x)dx \to 0 = f(1)$.

Now assume $f(1) > 0$. Assume we've taken ε smaller than $f(1)$.

Then
$$\lim_{n\to\infty} \int_{1-\delta}^1 nx^{n-1}f(x)dx \ge (f(1)-\varepsilon) \lim_{n\to\infty} \int_{1-\delta}^1 nx^{n-1}dx$$
$$= f(1) - \varepsilon.$$

Similarly, $\lim\limits_{n\to\infty} \int_{1-\delta}^1 nx^{n-1}f(x)dx \le f(1) + \varepsilon$.

Thus, since ε arbitrary, $\lim\limits_{n\to\infty} \int_{1-\delta}^1 nx^{n-1}f(x)dx = f(1)$.

This reasoning works when $f(1) < 0$ as well.

And $\lim\limits_{n\to\infty} \int_0^{1-\delta} nx^{n-1}f(x)dx = \int_0^{1-\delta} f(x) \lim\limits_{n\to\infty} nx^{n-1}dx = 0$ by DCT.

f continuous on $[0, 1-\delta]$ $\Rightarrow$ f has a maximum, say at x. Then,

$\frac{d}{dn} nx^{n-1} = x^{n-1} + nx^{n-1}h(x) = x^{n-1}(1 + nh(x)) = 0$ when $n = \frac{-1}{h(x)}$.

So for $n > \frac{-1}{h(x)}$, nx^{n-1} decreases.

Thus $\forall n$, $ny^{n-1}f(y) \le \left(\frac{-1}{h(x)}\right) x^{\left(-\frac{1}{h(x)} - 1\right)} f(x)$ which is integrable.

Hence, $\lim\limits_{n\to\infty} \int_0^1 nx^{n-1}f(x)dx = 0 + f(1) = f(1)$.

$\square$

17 sp.4

Problem 4. Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space. Let f, g be measurable real valued functions. Show that

$$\int |f-g| d\mu = \int_{-\infty}^{\infty} \int \left| \chi_{(t,\infty)}(f(x)) - \chi_{(t,\infty)}(g(x)) \right| d\mu(x) dt.$$

$$\begin{aligned}
&\int_{-\infty}^{\infty} \int | \chi_{(t,\infty)}(f(x)) - \chi_{(t,\infty)}(g(x)) | \, d\mu(x) \, dt \\
&= \int \int_{-\infty}^{\infty} | \chi_{(t,\infty)}(f(x)) - \chi_{(t,\infty)}(g(x)) | \, dt \, d\mu(x) \\
&= \int \int_{-\infty}^{\infty} | \chi_{(-\infty, f(x))}(t) - \chi_{(-\infty, g(x))}(t) | \, dt \, d\mu(x) \\
&= \int \int_{-\infty}^{\infty} \chi_{[\min\{f(x), g(x)\}, \max\{f(x), g(x)\})}(t) \, dt \, d\mu(x) \\
&= \int |f(x) - g(x)| \, d\mu
\end{aligned}$$

We can swap integrals because of Tonelli, since $|\chi_{(t,\infty)}(f(x)) - \chi_{(t,\infty)}(g(x))| \in L^+(\mathbb{R} \times \Omega)$.
Indeed, this function is measurable because the composition of measurable functions is measurable.

$\square$.