16 sp. 1

1. Let

$$f(y) = \sum_n \frac{x}{x^2 + yn^2}.$$

Show that $g(y) = \lim_{x\to\infty} f(x, y)$ exists for all $y > 0$. Find $g(y)$.

Kardás Doesn't work

Let $\varepsilon = \frac{1}{x}$. Then $f(y) = \sum_n \frac{1/\varepsilon}{1/\varepsilon^2 + yn^2} = \sum_n \frac{1}{\varepsilon(1 + yn^2\varepsilon^2)/\varepsilon^2}$

$= \sum_n \frac{\varepsilon}{1 + yn^2\varepsilon^2}$

$g(y) = \lim_{x\to\infty} f(x, y) = \lim_{\varepsilon\to 0} \sum_n \frac{\varepsilon}{1 + yn^2\varepsilon^2}$

$\frac{1}{1+yu^2} = g \qquad g(n\varepsilon)$

$\int_0^\infty \frac{1}{1+yu^2}\, dy$

16 sp. 2

2. Let $A \subseteq \mathbb{R}$ be Lebesgue measurable. Show that $n(\chi_A * \chi_{[0,\frac{1}{n}]}) \to \chi_A$ pointwise a.e. as $n \to \infty$. (Recall that $(f * g)(x) = \int f(x-y)g(y)\,dy$ for $x \in \mathbb{R}$.)

$$n(\chi_A * \chi_{[0,1/n]}) \longrightarrow \chi_A \qquad \text{as } n \to \infty$$

$$n \int \chi_A(x-y)\,\chi_{[0,1/n]}(y)\,dy \longrightarrow \chi_A(x)$$

$$= n \int_0^{1/n} \chi_A(x-y)\,dy$$

$$= n \int_0^{1/n} \chi_{x-A}(y)\,dy$$

$$= n\, m(x-A \cap [0,1/n]) \qquad \leq 1 \quad \forall n.$$

Assume not. Assume $\exists$ set E with $m(E) > 0$ and $n(\chi_A * \chi_{[0,1/n]}) \not\to \chi_A$ $\forall$ $x \in E$.

16 sp. 3

3. a) Prove that if a sequence of integrable functions f_n on $[0,1]$ satisfies $\int_0^1 |f_n(x)|\,dx \le 1/n^2$ for $n \in \mathbb{N}$, then $f_n \to 0$ a.e. on $[0,1]$ as $n \to \infty$.
b) Show that the above fact is not true if $1/n^2$ is replaced by $1/\sqrt{n}$.

a) Assume instead that $f_n \not\to 0$ a.e. on $[0,1]$ as $n \to \infty$.
Then $\exists$ set $E \subset [0,1]$ with $m(E) > 0$ s.t.
$f_n(x) \not\to 0$ $\forall$ $x \in E$. So for each $x \in E$, $\forall N \in \mathbb{N}$,
$\exists$ $n \ge N$ s.t. $f_n(x) \ge \varepsilon$, for a given $\varepsilon > 0$.

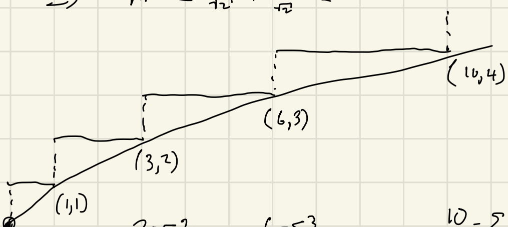
$$\int_0^1 |f_n(x)|\,dx \le 1/n^2 \Rightarrow \sum_{n=1}^\infty \int_0^1 |f_n(x)|\,dx \le \sum_{n=1}^\infty 1/n^2 < \infty$$
$$\Rightarrow \int_0^1 \sum_{n=1}^\infty |f_n(x)|\,dx < \infty$$

But for each $x \in E$, $\sum_{n=1}^\infty |f_n(x)| \ge \sum_1^\infty \varepsilon = \infty$
$\Rightarrow \int_0^1 \sum_{n=1}^\infty |f_n(x)|\,dx = \infty$ $\Rightarrow\Leftarrow$. $\square$

b) Consider the scanning function:
Define $g(k,l) := \chi_{[k/l,\, k+1/l]}$ for $l \in \mathbb{N}$, $k \in \{0, 1, \dots, l-1\}$.
Define $f_1 = g(0,1)$. For $f_n = g(k,l)$, define
$$f_{n+1} = \begin{cases} g(0, l+1) & \text{if } k = l-1 \\ g(k+1, l) & \text{otherwise.} \end{cases}$$

Thus, $\int_0^1 |f_n|\,dx = 1/l$ when $f_n = g(k,l)$.
We claim $\sqrt{n} \le l$ $\forall n$.
Indeed, $n \le \frac{l(l+1)}{2} = \frac{l^2}{2} + \frac{l}{2}$
$\Rightarrow$ $\sqrt{n} \le \frac{l}{\sqrt{2}} + \frac{\sqrt{l}}{\sqrt{2}} =$

$\int \lvert f_n \rvert$	$1/\sqrt{n}$
1	1
1/2	
1/2	
1/3	1/2
1/3	
1/3	
1/4	
1/4	
1/4	1/3
1/4	



$3 = \sum_1^2$ $\quad$ $6 = \sum_1^3$ $\quad$ $10 = \sum_1^4 = \frac{l(l+1)}{2}$

$y =$ $\quad$ $x = \sum_1^y n = \frac{y(y+1)}{2} = \frac{y^2+y}{2}$

16 sp.4

4. Let

$$f(x) = \begin{cases} \frac{1}{\sqrt{x}}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases} .$$

Also, let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the rationals. Define

$$g_n(x) = \frac{1}{2^n} f(x - r_n), \qquad x \in \mathbb{R}$$

and let

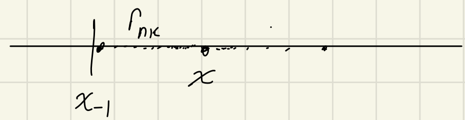
$$g(x) = \sum_{n=1}^{\infty} g_n(x), \qquad x \in \mathbb{R}$$

a) Prove that g is integrable on $\mathbb{R}$.
b) Prove that g is discontinuous at every point in $\mathbb{R}$.

a)
$$\begin{aligned} \int |g(x)|\, dx &= \int \sum_{n=1}^{\infty} g_n(x)\, dx \\ &= \int \sum_{n=1}^{\infty} \frac{1}{2^n} f(x - r_n)\, dx \\ &= \sum_{n=1}^{\infty} \frac{1}{2^n} \int f(x - r_n)\, dx \qquad \text{since } g_n \in L^+ \\ &= \sum_{n=1}^{\infty} \frac{1}{2^n} \int f(x)\, dx \\ &= 2 \sum_{n=1}^{\infty} \frac{1}{2^n} = 2 < \infty . \end{aligned}$$

b) Let $x \in \mathbb{R}$.

$$\begin{aligned} \lim_{y \to x} g(y) &= \lim_{y \to x} \sum_{n=1}^{\infty} \frac{1}{2^n} f(y - r_n) \\ &= \sum_{n=1}^{\infty} \frac{1}{2^n} \lim_{y \to x} f(y - r_n) \qquad \text{by DCT} \end{aligned}$$



Let x, r_n be fixed.

Assume $f(x - r_n) > 0$.

$y \to x$. $f(y - r_n) \to f(x - r_n)$. ✓

Assume $f(x - r_n) = f(0) = 0$

$y \nearrow x \Rightarrow f(y - r_n) \Rightarrow 0$

$y \searrow x \Rightarrow f(y - r_n) \to \infty$.