15 sp. 1

1. Consider the sequence

$$f_n(x) = \left(1+\frac{x}{n}\right)^{-n}\cos\left(\frac{x}{n}\right), \qquad n = 1, 2, \dots .$$

Evaluate

$$\lim_n \int_0^\infty f_n(x)\,dx,$$

being careful to justify your answer.

$$\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^{-n} = y$$

$$\lim_{n\to\infty} \ln\left[\left(1+\frac{x}{n}\right)^{-n}\right] = \ln y$$

$$\lim_{n\to\infty} -n\ln\left(1+\frac{x}{n}\right) = \ln y$$

$$\lim_{n\to\infty} -\frac{\ln\left(1+\frac{x}{n}\right)}{1/n} = \ln y$$

L'Hopital:

$$\lim_{n\to\infty} -\frac{\frac{n}{n+x}\left(-x/n^2\right)}{-1/n^2} = \ln y$$

$$\lim_{n\to\infty} -\frac{xn}{n+x} = \ln y$$

L'Hopital:

$$\lim_{n\to\infty} -\frac{x}{1} = \ln y \quad \Rightarrow \quad y = e^{-x}$$

$$\frac{x}{n} = x$$

$$1 = n$$

$$\frac{x}{n} < x \text{ for } n>1.$$

And $\lim_{n\to\infty} \cos\left(\frac{x}{n}\right) = \cos(0) = 1.$

Thus if DCT applies then $\lim_n \int_0^\infty f_n(x)\,dx = \int_0^\infty e^{-x}\,dx = -e^{-x}\Big]_0^\infty = 1.$

So let's examine $\left(1+\frac{x}{n}\right)^{-n}$. We can expand $\left(1+\frac{x}{n}\right)^n$:

$$\left(1+\frac{x}{n}\right)^n = 1 + n\left(\frac{x}{n}\right) + \binom{n}{2}\left(\frac{x}{n}\right)^2 + \cdots = 1 + x + \frac{n(n-1)}{2}\frac{x^2}{n^2} + \cdots = 1 + x + \frac{x^2(n-1)}{2n} + \cdots$$

$$\geq 1 + x + \frac{x^2(n-1)}{2n} \Rightarrow \left(1+\frac{x}{n}\right)^{-n} \leq \left(1+x+\frac{x^2(n-1)}{2n}\right)^{-1} \text{ for } n>2.$$

$$= \left(1+x+\frac{x^2}{2}-\frac{x^2}{2n}\right)^{-1}$$

$$\leq \left(1+x+\frac{x^2}{2}-\frac{x^2}{4}\right)^{-1}$$

$$= \left(1+x+\frac{x^2}{4}\right)^{-1}$$

$$\leq \left(\frac{x^2}{4}\right)^{-1} = \frac{4}{x^2} \in L^1. \quad \square$$

15 sp. 2

2. Suppose that $f\colon [0,\infty) \to \mathbb{R}$ is Lebesgue integrable.
(i) Show that there exists a sequence $x_n \to \infty$ such that $f(x_n) \to 0$.
(ii) Is it true that $f(x)$ must converge to 0 as $x \to \infty$? Give a proof or a counterexample.
(iii) Suppose additionally that $f$ is differentiable and $f'(x) \to 0$ as $x \to \infty$. Is it true that $f(x)$ must converge to 0 as $x \to \infty$? Give a proof or a counterexample.

(i) We can construct such a sequence. Assume $\nexists$ $x$ s.t. $|f(x)| < 1$. then $\int_0^\infty |f| dm \geq \int_0^1 dm = \infty \Rightarrow\Leftarrow$, so $\exists$ some $x_1$ with $f(x_1) < 1$. Now consider $(x_1, \infty)$ and apply the same logic to find some $x_2 \in (x_1, \infty)$ with $|f(x_2)| < \frac{1}{2}$. Then we find some $x_3 \in (x_2, \infty)$ with $|f(x_3)| < \frac{1}{3}$ and so on.

(ii) No. Consider $f(x) = \sum_1^\infty \chi_{[n, n+1/2^n]}$
$\int_0^\infty f\, dm = \sum_1^\infty m([n, n+1/2^n]) = \sum_1^\infty 1/2^n = 1 < \infty$.
But $f \not\to 0$ as $x \to \infty$.

(iii) Assume not. That is, $f \not\to 0$ as $x \to \infty$.
Then for some $\varepsilon > 0$, $\forall N \in \mathbb{N}$, $\exists x \geq N$ with $|f(x)| \geq \varepsilon$.
$f'(x) \to 0 \Rightarrow$ for some $M \in \mathbb{N}$, $\forall x \geq M$ $|f'(x)| < \varepsilon$.
There exists some $x_1 \geq M$ with $|f(x_1)| \geq \varepsilon$. Then $\int_{x_1}^{x_1+1} |f| dm \geq \varepsilon/2$ because $|f|$ cannot be less than the straight line function between the points of the graph $(x_1, \varepsilon)$ and $(x_1+1, 0)$. for if so then MVT would be violated.

We can continue to find points where this is true ad infinitum. Thus $\int_0^\infty |f| \geq \sum_1^\infty \varepsilon/2 = \infty$.

Hence, $f \to 0$ as $x \to \infty$.

15 sp. 3

3. Define $f_n(x) = ae^{-nax} - be^{-nbx}$ where $0 < a < b$.

(i) Show that

$$\sum_{n=1}^{\infty} \int_0^{\infty} f_n(x)\,dx = 0$$

and

$$\int_0^{\infty} \sum_{n=1}^{\infty} f_n(x)\,dx = \log(b/a).$$

(ii) What can you deduce about the value of

$$\int_0^{\infty} \sum_{n=1}^{\infty} |f_n(x)|\,dx?$$

(i) $\sum_{n=1}^{\infty} \int_0^{\infty} ae^{-nax} - be^{-nbx}\,dx = \sum_{n=1}^{\infty} -\frac{1}{n}e^{-nax} + \frac{1}{n}e^{-nbx}\Big]_0^{\infty}$

$= \sum_{n=1}^{\infty} 0 + 0 + \frac{1}{n}e^0 - \frac{1}{n}e^0$

$= \sum_{n=1}^{\infty} 0 = 0.$

$\sum_{n=1}^{\infty} ae^{-nax} = a\sum_{n=1}^{\infty}\left(\frac{1}{e^{ax}}\right)^n = a\left(\frac{1}{e^{ax}}\right)\Big/\left(1-\frac{1}{e^{ax}}\right) = \frac{ae^{-ax}}{1-e^{-ax}}$

$\Rightarrow \int_0^{\infty} \sum_{n=1}^{\infty} f_n(x)\,dx = \int_0^{\infty} \frac{ae^{-ax}}{1-e^{-ax}} - \frac{be^{-bx}}{1-e^{-bx}}\,dx$ $\qquad u = 1-e^{-ax}$, $du = ae^{-ax}\,dx$

$= \ln(1-e^{-ax}) - \ln(1-e^{-bx})\Big]_0^{\infty}$

$= \lim_{K\to\infty} \ln(1-e^{-ax}) - \ln(1-e^{-bx})\Big]_{1/K}^{\infty}$

$= \lim_{K\to\infty} \ln(1) - \ln(1) + \ln(1-e^{-b/K}) - \ln(1-e^{-a/K})$

$= \lim_{K\to\infty} \ln\left(\frac{1-e^{-b/K}}{1-e^{-a/K}}\right) = y$, say

$\lim_{K\to\infty} \frac{1-e^{-b/K}}{1-e^{-a/K}} = e^y$

L'Hopital $\Rightarrow$ $\lim_{K\to\infty} \frac{-b/K^2\, e^{-b/K}}{-a/K^2\, e^{-a/K}} = e^y$

$\Rightarrow \lim_{K\to\infty} \frac{b}{a} e^{(a-b)/K} = e^y$

$\Rightarrow b/a = e^y \Rightarrow y = \ln\left(\frac{b}{a}\right).$

(ii) $\int_0^{\infty} \sum_{n=1}^{\infty} |f_n(x)|\,dx = \infty$, for if not,
then $f_n(x) \in L^1(\nu \times m)$ where $\nu$ is the counting measure
$\Rightarrow$ by Fubini $\sum_{n=1}^{\infty} \int_0^{\infty} f_n(x)\,dx = \int_0^{\infty} \sum_{n=1}^{\infty} f_n(x)\,dx$, a contradiction.

$\square$

15 sp. 4

4. Assume that $f$ is integrable on $[0,1]$ with respect to the Lebesgue measure $m$, and let $F(x) = \int_0^x f(t)\,dt$. Assume that $\phi\colon \mathbb{R} \to \mathbb{R}$ is Lipschitz, i.e., there exists a constant $C \geq 0$ such that

$$|\phi(x_1) - \phi(x_2)| \leq C|x_1 - x_2|, \qquad x_1, x_2 \in \mathbb{R}.$$

Prove that there exists a function $g$ which is integrable on $[0,1]$ such that $\phi(F(x)) = \int_0^x g(t)\,dt$ for $x \in [0,1]$.

Assuming $\phi(0) = 0$

The Fundamental Thm of Calc. for Lebesgue Integrals tells us that $\phi(F(x))$ is absolutely continuous on $[0,1] \iff \exists\, g \in L^1([0,1])$ s.t. $\int_0^x g(t)\,dt = \phi(F(x)) - \phi(F(0)) = \phi(F(x)) - \phi(0) = \phi(F(x))$.

So we need to show $\phi(F(x))$ absolutely continuous.

F is absolutely continuous by this theorem.

Let $\varepsilon > 0$. $\exists\, \delta$ s.t. for a collection of disjoint intervals on $[0,1]$, $\{(a_i, b_i)\}_1^N$ we have

$$\sum_1^N (b_i - a_i) < \delta \Rightarrow \sum_1^N |F(b_i) - F(a_i)| < \varepsilon / C^*$$

$$\phi \text{ Lipschitz} \Rightarrow \sum_1^N |\phi(F(b_i)) - \phi(F(a_i))| \leq \sum_1^N C\,|F(b_i) - F(a_i)| < C\,\varepsilon/C = \varepsilon.$$

Thus $\phi(F(x))$ is absolutely continuous as well.

□

* Note if $C = 0$ then $\phi(F(x)) = F(x)$ and we are done.