(1)

$$f_n(x) = \left(1 + \frac{x}{n}\right)^{-n} \cos\left(\frac{x}{n}\right)$$

Evaluate $\lim_n \int_0^\infty f_n \, dx$. real Spring 2015

$$\int_0^\infty |f_n| = \int_0^\infty \frac{dx}{(1+\frac{x}{n})^n} = n\int_1^\infty \frac{du}{u^n} < \infty \quad \text{for } n > 2.$$

so $\int_0^\infty f_n$ makes sense.

Also, $f_n(x) \longrightarrow e^{-x}$ pointwise.

For $x > 0$,

$$\left|1 + \frac{x}{n}\right|^n \geq 1 + x + \frac{\cancel{n}(n-1)}{2n} \frac{x^2}{\cancel{n^2}} \qquad \frac{n-1}{2n} \geq \frac{1}{2}\cdot\frac{1}{2} \; (n \geq 2)$$

$$\Rightarrow \frac{1}{\left|1 + \frac{x}{n}\right|^n} \leq \frac{1}{1 + x + \frac{n-1}{2n}x^2}$$

$$\leq \frac{1}{1 + x + \frac{1}{4}x^2} = \frac{4}{4 + 4x + x^2}$$

which is in $L^1([0,\infty))$.

so by the DCT,

$$\lim_{n\to\infty} \int_0^\infty \left(1 + \frac{x}{n}\right)^{-n} \cos\left(\frac{x}{n}\right) dx = \int_0^\infty e^{-x} dx = 1.$$

$\blacksquare$

(2) $f : [0,\infty) \to \mathbb{R}$ Lebesgue integrable.

(i) Show there is a sequence x_n s.t. $f(x_n) \to 0$.

solution

If $\{x : |f(x)| \leq \varepsilon\}$ had measure 0,

then $\int |f| \geq \varepsilon m[0,\infty] = \infty$. So $m(\{x : |f(x)| \leq \varepsilon\}) > 0$

$\Rightarrow$ it is nonempty $\Longrightarrow$ pick $x_n \in \{x : |f(x)| \leq \frac{1}{n}\}$.

$\Rightarrow$ $f(x_n) \to 0$. $\square$

(ii) Is it true that $f(x)$ must converge to 0 as $x \to \infty$?

solution

No. $f(x) = \begin{cases} 1, x \in \mathbb{N} \\ 0, \text{otherwise} \end{cases}$ $\int f = 0$.

(iii) Suppose f is differentiable and $f'(x) \to 0$ as $x \to \infty$. Is it true $f(x) \to 0$ as $x \to \infty$?

solution Yes. Assume $|f(x_n)| \geq 2\varepsilon$ for $x_n \leq x_n + 1 \leq x_{n+1} \leq x_{n+1} + 1 \leq \cdots$

Take x_n large enough s.t. $|f'(x_n)| \leq \frac{\varepsilon}{2}$.

$\longrightarrow$

Then on $[x_n, x_n+1]$, $\left| f(x) - f(x_n) \right| = \left| \int_{x_n}^{x} f' \right|$

$$\le \int_{x_n}^{x_n+1} |f'| \le \frac{\varepsilon}{2} \cdot 1.$$ Which means on

$[x_n, x_n+1]$, $|f(x)| \geq \varepsilon$. Consequently,

$$\int |f| \geq \sum \int_{x_n}^{x_n+1} |f| \geq \sum^{\infty} \varepsilon = \infty. \Rightarrow\!\Leftarrow.$$

(3) $\quad f_n(x) = a e^{-nax} - b e^{-nbx}, \quad 0 < a < b.$

(i) Show $\displaystyle\sum_{n=1}^{\infty} \int_0^{\infty} f_n(x)\,dx = 0$

and $\displaystyle\int_0^{\infty} \sum_{n=1}^{\infty} f_n(x)\,dx = \log(b/a).$

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e^{-ax} and $e^{-bx} < 1$ for $x > 0$.

$$\int_0^{\infty} \sum_{n=1}^{\infty} a e^{-nax} - b e^{-nbx}$$

$|x|<1 \Rightarrow \displaystyle\sum_{n=1}^{\infty} x^n = \frac{1}{1-x} - 1 = \frac{1-(1-x)}{1-x} = \frac{x}{1-x}.$

$$= \int_0^{\infty} \frac{a e^{-ax}}{1-e^{-ax}} - \frac{b e^{-bx}}{1-e^{-bx}} = \cancel{\ln(1-e^{-ax}) - \ln(1-e^{-bx})\Big|_0^{\infty}}$$

$$= \int_0^{\infty} \frac{a e^{-ax}(1-e^{-bx}) - b e^{-bx}(1-e^{-ax})}{(1-e^{-ax})(1-e^{-bx})}\,dx$$

(ii) $\displaystyle\int_0^{\infty} \sum_{n=1}^{\infty} |f_n(x)|\,dx = \infty$. For otherwise, by Fubini's theorem, the two values $\sum\int$ & $\int\sum$ would coincide.

(4) f integrable on $[0,1]$, w.r.t. m.

$F(X) = \int_0^X f(t)\,dt$. $\phi : \mathbb{R} \to \mathbb{R}$ Lipschitz.

Prove there is g integrable on $[0,1]$ s.t.

$$\phi(F(X)) = \int_0^X g(t)\,dt \text{ for } X \in [0,1].$$

$F(0)=0$. Since $|\phi(0)-\phi(\varepsilon)| \le C|\varepsilon|$, as $\varepsilon \to 0$ we find $\phi(0)=0$. ϕ is also absolutely continuous. So ϕ is in BV. Since $\phi(0)=0$, ϕ is in NBV. Moreover, since $F(0)=0$, $\phi(F)$ is in NBV. So we obtain a measure $\mu_{\phi(F)} \ll m$ (since $\phi(F)$ absolutely continuous) s.t. $\mu_{\phi(F)}((-\infty, x]) = \phi(F(X))$. Let g be $\frac{d\mu_{\phi(F)}}{dm}$. Then by finiteness $g \in L^1$. And

$\int_0^x g\,dm =$ [illegible] $=$ $d\mu_{\phi(F)}([0,x]) = \phi(F(X))$ (since there is no mass on $[-\infty,0]$).