14 Sp. 1

1. Suppose that $(X, \mathcal{B}, \mu)$ is a measure space with $\mu(X) < \infty$, and that $\{f_n\}_{n\geq 1}$ and f are measurable functions on X such that $f_n \to f$ almost everywhere.

(i) Suppose that $\int f^2\, d\mu < \infty$. Show that f is integrable.

(ii) Suppose that there exists $C < \infty$ such that $\int f_n^2\, d\mu \leq C$ for all $n \geq 1$. Show that $f_n \to f$ in L^1.

(iii) Give an example where $\int |f_n|\, d\mu \leq 1$ for all $n \geq 1$ but $f_n \not\to f$ in L^1.

(i) $$\int f^2 d\mu = \int_{\{x:\, |f(x)| \leq 1\}} f^2 d\mu + \int_{\{x:\, |f(x)| > 1\}} f^2 d\mu$$
$$\Rightarrow \int_{\{x:\, |f(x)|>1\}} f^2 d\mu < \infty$$
$$\Rightarrow \int |f|\, d\mu = \int_{\{x:\, |f(x)| \leq 1\}} |f|\, d\mu + \int_{\{x:\, |f(x)|>1\}} |f|\, d\mu \leq \mu(X) + \int_{\{x:\, |f(x)|>1\}} f^2 d\mu < \infty. \quad \square$$

(ii) NTS $\int |f_n - f|\, d\mu \to 0$ as $n \to \infty$.

?

why is $|f_n(x) - f(x)| \leq 2\,|f(x)|$ a.e.

$M \quad \int_{\{x:\, f > M\}} f_n$

$M f_n \chi_{\{x:\, f_n \geq M\}} \leq f_n^2$ $\quad \cancel{*}$

(iii) Let $f_n = n\chi_{[0, 1/n]}$, $f_n \to 0$ a.e. But $\int |f_n - 0|\, d\mu = 1 \quad \forall n$.
So $f_n \not\to 0$ in L^1. $\square$

14Sp. 2

2. For what non-negative integer n and positive real c does the integral

$$\int_1^\infty \ln\left(1 + \frac{(\sin x)^n}{x^c}\right) dx$$

(a) exist as a (finite) Lebesgue integral? only depends on c.

(b) converge as an improper Riemann integral?

(a)

Is Kayla

$\ln(1+y) \approx y$ $\longrightarrow \int \frac{(\sin x)^n}{x^c}\,dx$

$\ln(1+y) \le 2y$

$\boxed{\begin{array}{ll} c > 1. & \text{integrable} \\ c \le 1 & \text{not.} \end{array}}$

$\{x : \sin x \ge \frac{1}{2}\}$

bound $\frac{(\sin x)^n}{x^c}$ below by $\frac{1}{x^c}$

pieces of intervals of length 2π.

14 Sp. 3

3. Suppose f is Lebesgue integrable on $\mathbb{R}$. Show that

$$\lim_{t\to 0}\int_{-\infty}^{\infty}|f(x+t)-f(x)|dx = 0.$$

First consider a continuous function g that vanishes outside $[-M, M]$.

Do.

14 Sp. 4

4. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces such that $\mu(X) > 0$ and $\nu(Y) > 0$. Let $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ be measurable functions (with respect to $\mathcal{A}$ and $\mathcal{B}$ respectively) such that

$$f(x) = g(y) \quad \mu \times \nu \text{ -almost everywhere on } X \times Y$$

Show that there exists a constant λ such that $f(x) = \lambda$ for μ-a.e. x and $g(y) = \lambda$ for ν-a.e. y.

Assume $\exists\ \lambda \in \mathbb{R}$ s.t. $\mu(f^{-1}(\lambda)) > 0$.

$f(x) = g(y)$ $\mu \times \nu$ - a.e.

$\Rightarrow \mu \times \nu\left(f^{-1}(\lambda) \times g^{-1}(\mathbb{R} \setminus \{\lambda\})\right) = \mu(f^{-1}(\lambda)) \cdot \nu(g^{-1}(\mathbb{R} \setminus \{\lambda\})) = 0$

$\Rightarrow \nu(g^{-1}(\mathbb{R} \setminus \{\lambda\})) = 0 \Rightarrow g(y) = \lambda$ ν-a.e.

$\Rightarrow \nu(g^{-1}(\lambda)) = \nu(Y) > 0$

$\Rightarrow$ by the same logic, $f(x) = \lambda$ μ-a.e.

We get the same result if our starting assumption is that $\exists\ \lambda \in \mathbb{R}$ s.t. $\nu(g^{-1}(\lambda)) > 0$.

So, what happens if $\nexists\ \lambda$ s.t. $\mu(f^{-1}(\lambda)) > 0$ or $\nu(g^{-1}(\lambda)) > 0$? That is, $\forall\ \lambda \in \mathbb{R}$, $\mu(f^{-1}(\lambda)) = 0$ and $\nu(g^{-1}(\lambda)) = 0$.

Let $E \subset \mathbb{R}$ be a measurable subset.

$$\mu \times \nu\left(f^{-1}(E) \times g^{-1}(E^c)\right) = 0. \qquad \forall\ E \subset \mathbb{R}.$$

$\mu(f^{-1}(E)) > 0$ for some $E \subset \mathbb{R} \Rightarrow \nu(g^{-1}(E^c)) = 0$

$\Rightarrow \nu(g^{-1}(E)) = \nu(Y) \Rightarrow \mu(f^{-1}(E^c)) = 0$

$\Rightarrow \mu(f^{-1}(E)) = \mu(X)$.

$\mu(f^{-1}(\mathbb{R})) > 0 \Rightarrow \mu(f^{-1}(\mathbb{R})) = \mu(X)$. ?

$\mu(f^{-1}(\mathbb{R})) = \mu(f^{-1}((-\infty, 0))) + \mu(f^{-1}([0, \infty))) = \mu(X)$

$\Rightarrow$ one of $= \mu(X)$.