13 Sp. 1

1. Suppose that $\{f_n\}$ is a sequence of of real valued continuously differentiable functions on $[0,1]$ such that

$$\lim_{n\to\infty}\int_0^1 |f_n(x)|\,dx = 0 \text{ and } \lim_{n\to\infty}\int_0^1 |f_n'(x)|\,dx = 0.$$

Show that $\{f_n\}$ converges to 0 uniformly on $[0,1]$.

Let $\varepsilon > 0$.

Let N be large enough so that $\forall\, n \geq N$,

$$\int_0^1 |f_n(x)|\,dx < \varepsilon/2 \text{ AND } \int_0^1 |f_n'(x)|\,dx < \varepsilon/2.$$

Assume $\exists\, n \geq N$, $x \in [0,1]$ s.t. $|f_n(x)| \geq \varepsilon$.

$\forall\, y \leq x \in [0,1]$, we have

$$|f_n(x) - f_n(y)| = \left|\int_y^x f_n'(t)\,dt\right| \leq \int_y^x |f_n'(t)|\,dt < \varepsilon/2.$$

and $\forall\, y \geq x \in [0,1]$, we have

$$|f_n(y) - f_n(x)| = \left|\int_x^y f_n'(t)\,dt\right| \leq \int_x^y |f_n'(t)|\,dt < \varepsilon/2.$$

$\Rightarrow |f_n(y)| > \varepsilon/2 \quad \forall\, y \in [0,1]$.

$\Rightarrow \int_0^1 |f_n(x)|\,dx > \varepsilon/2$, a contradiction.

Thus, assumption is wrong.

$\Rightarrow \forall\, n \geq N,\ x \in [0,1],\quad |f_n(x)| < \varepsilon.$

Hence, we have shown $f_n \to 0$ uniformly on $[0,1]$.

$\square$

13 Sp. 2

2. Investigate the convergence of $\sum_{n=0}^{\infty} a_n$, where

$$a_n = \int_0^1 \frac{x^n}{1-x} \sin(\pi x)\, dx.$$

$$\sum_{n=0}^{\infty} a_n = \lim_{N \to \infty} \sum_{n=0}^{N} a_n = \lim_{N\to\infty} \sum_{n=0}^{N} \int_0^1 \frac{x^n}{1-x} \sin(\pi x)\, dx$$

$$= \lim_{N\to\infty} \int_0^1 \sum_{n=0}^{N} \frac{x^n}{1-x} \sin(\pi x)\, dx$$

* DCT =>

$$= \int_0^1 \sum_{n=0}^{\infty} \frac{x^n}{1-x} \sin(\pi x)\, dx$$

$$= \int_0^1 \frac{1}{(1-x)^2} \sin(\pi x)\, dx$$

$$\sin(\pi x) = (\pi x) - \frac{(\pi x)^3}{3!} + \frac{(\pi x)^5}{5!} - \cdots$$

Sp 13.3

3. Let $(X, \mathcal{M}, \mu)$ be a measure space, $f_n, f \in L^1(\mu)$. Show that $\int_X |f_n - f|\, d\mu \to 0$ as $n \to \infty$ if and only if

$$\sup_{A\in\mathcal{M}} \left| \int_A f_n d\mu - \int_A f d\mu \right| \to 0$$

as $n \to \infty$.

$(\Rightarrow)$:

$\forall A \in \mathcal{M}$, $\quad \left| \int_A f_n \, d\mu - \int_A f \, d\mu \right| = \left| \int_A f_n - f \, d\mu \right|$

$$\leq \int_A |f_n - f| \, d\mu$$

$$\leq \int_X |f_n - f| \, d\mu$$

$\Rightarrow \sup_{A\in\mathcal{M}} \left| \int_A f_n \, d\mu - \int_A f \, d\mu \right| \leq \int_X |f_n - f| \, d\mu \to 0$ as $n \to \infty$. $\square$

$(\Leftarrow)$: $\sup_{A\in\mathcal{M}} \left| \int_A f_n \, d\mu - \int_A f \, d\mu \right| \to 0$ as $n \to \infty$

$\Rightarrow \exists \ N$ s.t. $\forall \ n \geq N$, $\sup_{A\in\mathcal{M}} \left| \int_A f_n \, d\mu - \int_A f \, d\mu \right| < \varepsilon$.

$\Rightarrow \left| \int_{\{x: f_n > f\}} f_n \, d\mu - \int_{\{x: f_n > f\}} d\mu \right|$

$= \left| \int_{\{x: f_n > f\}} f_n - f \, d\mu \right| = \int_{\{x: f_n > f\}} f_n - f \, d\mu < \varepsilon$.

And $\left| \int_{\{x: f_n \leq f\}} f_n - f \, d\mu \right| = \int_{\{x: f_n \leq f\}} f - f_n \, d\mu < \varepsilon$.

$\Rightarrow \int_X |f_n - f| \, d\mu = \int_{\{x: f_n > f\}} f_n - f \, d\mu + \int_{\{x: f_n \leq f\}} f - f_n \, d\mu$

$$< 2\varepsilon.$$

Thus $\int_X |f_n - f| \, d\mu \to 0$ as $n \to \infty$. $\square$

13 Sp. 4

4. Let μ and ν be σ-finite positive measures, $\mu \geq \nu$ and assume that $\nu << \mu - \nu$ (ν is absolutely continuous with respect to $\mu - \nu$).

Prove that

$$\mu\left(\{\frac{d\nu}{d\mu} = 1\}\right) = 0.$$

$$\mu(\star) > 0$$

$$\nu(\star) = \int_\star d\nu = \int_\star \frac{d\nu}{d\mu}\, d\mu = \mu(\star)$$