Spring 13 :

① $(f_n) \subseteq C^1([0,1])$ s.t. $\lim_{n\to\infty}\int_0^1 |f_n(x)|dx = 0$ (1) and $\lim_{n\to\infty}\int_0^1 |f_n'(x)|dx = 0$ (2)

Show $f_n \to 0$ uniformly :

Want to show that for $\varepsilon > 0$, $\exists N \geq 0$ s.t. $n \geq N \Rightarrow |f_n(x)| < \varepsilon \ \forall x$.

First see that for $\frac{\varepsilon}{2} > 0$, $\exists N \geq 0$ s.t. $n \geq N \Rightarrow \int_0^1 |f_n'(x)|dx < \frac{\varepsilon}{2}$, namely that, for $a, b \in [0,1]$:

$$|f_n(b) - f_n(a)| = \left|\int_a^b f_n'(x)\right| \leq \int_a^b |f_n'(x)|dx \leq \int_0^1 |f_n'(x)|dx < \frac{\varepsilon}{2} \quad \text{use (2)}$$

Therefore, $|f_n(0) - f_n(x)| < \frac{\varepsilon}{2}$ for all $x \in [0,1]$, hence

$$f_n(0) - \frac{\varepsilon}{2} < |f_n(x)| < f_n(0) + \frac{\varepsilon}{2}$$

$$\Rightarrow f_n(0) < |f_n(x)| + \frac{\varepsilon}{2} \Rightarrow \int_0^1 f_n(0)\,dx < \int_0^1 \left(|f_n(x)| + \frac{\varepsilon}{2}\right)$$

$$\Rightarrow f_n(0) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \text{use (1)}$$

Hence $f_n(0) \to 0$ as $n \to \infty$, i.e. for $\frac{\varepsilon}{2} > 0$, $\exists M$ s.t. $n \geq M$ implies

$$|f_n(x)| < f_n(0) + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{for all } x.$$

Thus $f_n \to 0$ unif.

② Determine the convergence of $\sum_{n=0}^{\infty} \int_0^1 \frac{x^n \sin(\pi x)}{(1-x)^2}dx$

First see that $\sum_{n=0}^{\infty} \int \frac{\sin \pi x}{(1-x)^2}dx = \int_0^1 \sum_{n=0}^{\infty} \frac{x^n \sin(\pi x)}{(1-x)^2}dx$ by Tonelli.

$$= \int_0^1 \frac{\sin(\pi x)}{(1-x)^2}dx \quad \text{by geometric sum}$$

Then, by substitution $u = 1-x$:

$$\int_0^1 \frac{\sin(\pi x)}{(1-x)^2}dx = \int_1^0 \frac{-\sin(\pi(1-u))du}{u^2} = \int_0^1 \frac{\sin(\pi(1-u))}{u^2}du = \int_0^1 \frac{\sin(\pi u)}{u^2}du$$

by symmetry of $\sin(\pi u)$ over $[0,1]$

$= \pi\int_0^1 \frac{\sin(\pi u)}{\pi u^2}du$; recall that $\lim_{u\to 0} \frac{\sin(\pi u)}{\pi u} = 1$,

hence we may choose δ s.t. $|u| < \delta \Rightarrow \left|\frac{\sin \pi u}{\pi u} - 1\right| < \frac{1}{2}$,

i.e. $-\frac{1}{2} < \frac{\sin \pi u}{\pi u} - 1 < \frac{1}{2} \Rightarrow \frac{1}{2} < \frac{\sin \pi u}{\pi u} < \frac{3}{2}$

$$\Rightarrow \pi\int_0^1 \frac{\sin(\pi u)}{\pi u}\cdot\frac{du}{u} > \pi\int_\delta^1 \frac{1}{2}\frac{du}{u} = \frac{\pi}{2}\int_\delta^1 \frac{du}{u} = \infty$$

as $\delta \to 0$.

hence divergent

③ (X,μ) meas, $f_n, f \in L^1(\mu)$. Show

$$\int_X |f_n - f| \, d\mu \to 0 \text{ as } n\to\infty \iff \sup_{A\in M_\mu} \left|\int_A f_n \, d\mu - \int_A f \, d\mu\right| \to 0 \text{ as } n\to\infty$$

$(\Rightarrow)$ $\varepsilon > 0$, $\exists N$ s.t. $n \geq N \Rightarrow \int_X |f_n - f| \, d\mu < \varepsilon$

Hence: $\sup_{A\in M_\mu} \left|\int_A f_n - \int_A f\right| \leq \sup_{A\in M_\mu} \int_A |f_n - f| \leq \int_X |f_n - f| < \varepsilon$,

thus $\sup_{A\in M_\mu} \left|\int_A f_n - \int_A f\right| \to 0$ as $n \to \infty$

$(\Leftarrow)$ $\varepsilon > 0$, $\exists N$ st. $n \geq N \Rightarrow \sup_{A\in M_\mu} \left|\int_A (f_n - f) \, d\mu\right| < \varepsilon$.

Now: $$\int_X |f_n - f| \, d\mu = \int_{\{f_n - f \geq 0\}} |f_n - f| \, d\mu + \int_{\{f_n - f < 0\}} |f_n - f| \, d\mu$$
$$= \left|\int_{\{f_n - f \geq 0\}} (f_n - f) \, d\mu\right| + \left|\int_{\{f_n - f < 0\}} (f_n - f) \, d\mu\right|$$
$$\leq \sup_{A\in M_\mu} \left|\int_A (f_n - f) \, d\mu\right| + \sup_{A\in M_\mu} \left|\int_A (f_n - f) \, d\mu\right|$$
$$= 2 \sup_{A\in M_\mu} \left|\int_A (f_n - f) \, d\mu\right| < 2\varepsilon$$

Hence $\int_X |f_n - f| \, d\mu \to 0$ as $n \to \infty$.

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④ μ, ν σ-finite, $\mu \geq \nu$, $\nu \ll \mu - \nu$. Claim: $\mu\left(\left\{\frac{d\nu}{d\mu} = 1\right\}\right) = 0$

First, see that $\mu(E) = 0 \Rightarrow \nu(E) \leq \mu(E) = 0 \Rightarrow \nu(E) = 0$ since ν is positive, hence the derivative $\frac{d\nu}{d\mu}$ is valid.

Now, $\nu \ll \mu - \nu$, hence $d\nu = g \, d(\mu - \nu)$ for some g. Hence:

$\nu(E) = \int_E g \, d\mu - g \, d\nu$. Then:

$$\nu\left(\left\{\frac{d\nu}{d\mu} = 1\right\}\right) = \int_{\left\{\frac{d\nu}{d\mu} = 1\right\}} d\nu = \int_{\left\{\frac{d\nu}{d\mu} = 1\right\}} \frac{d\nu}{d\mu} \, d\mu = \int_{\left\{\frac{d\nu}{d\mu} = 1\right\}} d\mu = \mu\left(\left\{\frac{d\nu}{d\mu} = 1\right\}\right),$$

but:

$$\nu\left(\left\{\frac{d\nu}{d\mu} = 1\right\}\right) = \int_{\left\{\frac{d\nu}{d\mu} = 1\right\}} g \, d\mu - g \, d\nu = \int_{\left\{\frac{d\nu}{d\mu} = 1\right\}} g \, d\mu - g \frac{d\nu}{d\mu} \, d\mu$$

$= \int_{\left\{\frac{d\nu}{d\mu} = 1\right\}} g \, d\mu - g \, d\mu = 0$, ie. $0 = \nu\left(\left\{\frac{d\nu}{d\mu} = 1\right\}\right) = \mu\left(\left\{\frac{d\nu}{d\mu} = 1\right\}\right)$