Real, Spring 2013

(1) $\{f_n\}$ real-valued, continuously differentiable on $[0,1]$

s.t. $\lim_{n\to\infty} \int_0^1 |f_n| = 0$ and $\lim_{n\to\infty} \int_0^1 |f_n'| = 0$.

Show $f_n \to 0$ uniformly on $[0,1]$.

Fix $\varepsilon > 0$. For any $x, y \in [0,1]$, we have

$$|f_n(x) - f_n(y)| = \left| \int_x^y f_n' \right| \leq \int_x^y |f_n'|$$
$$\leq \int |f_n'| \leq \varepsilon$$

for n sufficiently large.

Also, $m(\{x : |f_n(x)| \geq \varepsilon\}) \leq \varepsilon$ for n sufficiently large, for otherwise $\int |f_n| \geq \varepsilon^2$ for arbitrarily large n, violating $\int |f_n| \longrightarrow 0$.

Taking ε small, $n \gg 1$, let $|f_n(x_n)| \leq \varepsilon$.

By above, for all $x \in [0,1]$, $|f_n(x)| \leq |f_n(x_n)| + \varepsilon \leq 2\varepsilon$. Since n is arbitrary, $f_n \to 0$ uniformly on $[0,1]$. $\square$

(2) Investigate the convergence of $\sum_{n=0}^{\infty} a_n$ where $a_n = \int_0^1 \frac{x^n}{1-x} \sin(\pi x)\, dx$.

By Tonelli's theorem, since $\frac{x^n}{1-x}\sin(\pi x) \geq 0$ for $n \geq 1$, $x \in [0,1]$, we have

$$\sum_{n=0}^{\infty} \int_0^1 \frac{x^n}{1-x} \sin(\pi x)\, dx = \int_0^1 \sum_{n=0}^{\infty} \frac{x^n}{1-x} \sin(\pi x)\, dx$$

$$= \int_0^1 \frac{\sin(\pi x)}{1-x} \sum_{n=0}^{\infty} x^n \underset{x<1}{=} \int_0^1 \frac{\sin \pi x}{(1-x)^2}. \quad \cancel{\leq \int_0^1 \frac{dx}{(1-x)^2}}$$

$\cancel{= \int_0^1 \frac{dx}{x^2}}$ And $\int_0^1 \frac{\sin \pi x}{(1-x)^2} = \infty$.

Indeed, it equals $\int_0^1 \frac{\sin \pi(1-x)}{x^2} = \pi^2 \int_0^1 \frac{\sin \pi x}{(\pi x)^2} = \pi \int_0^{\pi} \frac{\sin x}{x^2}$;

we show $\int_0^{\pi} \frac{\sin x}{x^2} = \infty$. Assume it were finite.

Since $\frac{\sin x}{x} \to 1$ from below, for $\varepsilon > 0$

choose a small $\delta > 0$ s.t. $\frac{\sin x}{x} \geq 1 - \varepsilon$ for $0 < x < \delta$.

Then $\displaystyle\int_0^\delta \frac{\sin x}{x^2} \geq (1-\varepsilon)\int_0^\delta \frac{dx}{x} = \infty$ $\Rightarrow\!\Leftarrow$.

We conclude $\displaystyle\sum_{n=0}^{\infty} a_n = \infty$. $\square$

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(3) $f_n, f \in L^1(\mu, X)$. Show

$$\int |f_n - f| \to 0 \quad \text{iff} \quad \sup_{E \in \mathcal{M}} \left| \int_E f_n - \int_E f \right| \to 0.$$

Fix $\varepsilon > 0$. For sufficiently large n,

$$\left| \int_E f_n - \int_E f \right| \leq \int |f_n - f| \leq \varepsilon$$

$$\Rightarrow \sup_{E \in \mathcal{M}} \left| \int_E f_n - \int_E f \right| \leq \varepsilon. \quad ✓$$

For the other direction, let $E_n = \{x : f_n(x) \geq f(x)\}$ and $F_n = \{x : f_n(x) < f(x)\}$.

Then $$\int |f-f_n| = \int_{E_n} f_n - f + \int_{F_n} f - f_n$$

$$= \int_{E_n} f_n - f + \left| \int_{F_n} f_n - \int_{F_n} f \right|$$

$$\leq 2 \sup_{E \in \mathfrak{M}} \left| \int_E f_n - \int_E f \right| \longrightarrow 0. \quad \square$$

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(4) μ, ν σ-finite positive measures.

$\mu \geq \nu$ and $\nu \ll \mu - \nu$.

Prove $\mu\left(\left\{\frac{d\nu}{d\mu} = 1\right\}\right) = 0$.

$$\nu\left(\left\{\frac{d\nu}{d\mu} = 1\right\}\right) = \int_{\left\{\frac{d\nu}{d\mu}=1\right\}} 1 \, d\mu = \mu\left(\left\{\frac{d\nu}{d\mu}=1\right\}\right).$$

Assume $\mu < \infty \Rightarrow \nu < \infty$. Then, subtracting,

$$\mu\left(\left\{\frac{d\nu}{d\mu}=1\right\}\right) - \nu\left(\left\{\frac{d\nu}{d\mu}=1\right\}\right) = 0 \xRightarrow{\nu \ll \mu - \nu} \nu\left(\left\{\frac{d\nu}{d\mu}=1\right\}\right) = 0$$

$$\| $$

$$\mu\left(\left\{\frac{d\nu}{d\mu}=1\right\}\right). \checkmark$$

Extending to μ σ-finite is trivial. $X = \bigcup X_j$ disjoint $\mu(X_j) < \infty$. On X_j, $\int_E \frac{d\nu}{d\mu}\Big|_{X_j} d\mu_j = \int_E \frac{d\nu}{d\mu} d\mu = \nu(E) = \nu_j(E) \Rightarrow \frac{d\nu}{d\mu}\Big|_{X_j} = \frac{d\nu_j}{d\mu_j}$

$\Rightarrow \left\{x : \frac{d\nu}{d\mu} = 1\right\} = \bigcup_j \left\{x : \frac{d\nu_j}{d\mu_j} = 1\right\}$. Then since $\mu_j\left(\left\{x : \frac{d\nu_j}{d\mu_j} = 1\right\}\right) = 0 \Rightarrow \mu\left(\left\{x : \frac{d\nu}{d\mu} = 1\right\}\right) = \sum 0 = 0$. $\square$