12 Sp. 1

1. Let f and g be real integrable functions on a σ-finite measure space $(X, \mathcal{M}, \mu)$, and for $t \in \mathbb{R}$ let

$$F_t = \{x \in \cancel{E}\, X : f(x) > t\} \quad \text{and} \quad G_t = \{x \in \cancel{E}\, X : g(x) > t\}.$$

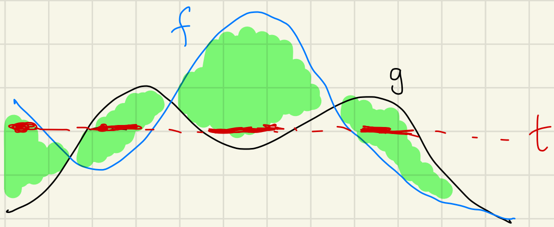
Show that

$$\int_X |f-g|\,d\mu = \int_{-\infty}^{\infty} \mu\left((F_t \backslash G_t) \cup (G_t \backslash F_t)\right) dt.$$

$$F_t \setminus G_t = \{x \in X : g(x) \le t < f(x)\}$$
$$G_t \setminus F_t = \{x \in X : f(x) \le t < g(x)\}$$ Disjoint.

$$\Rightarrow \mu((F_t \setminus G_t) \cup (G_t \setminus F_t)) = \mu(F_t \setminus G_t) + \mu(G_t \setminus F_t).$$

$$LHS = \int_{f>g} f-g\,d\mu + \int_{g>f} g-f\,d\mu$$



NTS $\int_{f>g} f-g\,d\mu = \int_{-\infty}^{\infty} \mu(F_t \setminus G_t)\,dt$

$$\mu(F_t \setminus G_t) = \int_X \chi_{F_t \setminus G_t}\,d\mu$$

$$\Rightarrow \int_{-\infty}^{\infty} \mu(F_t \setminus G_t)\,dt = \int_{-\infty}^{\infty}\int_X \chi_{F_t\setminus G_t}\,d\mu\,dt$$

$$= \int_X \int_{-\infty}^{\infty} \chi_{F_t \setminus G_t}\,dt\,d\mu \qquad \text{by Fubini}$$

$$= \int_{f>g} \int_{g(x)}^{f(x)} dt\,d\mu$$

$$= \int_{f>g} f-g\,d\mu.$$

12 Sp. 2

2. Show that

$$\int_{\pi}^{\infty} \frac{dx}{x^2(\sin^2 x)^{1/3}}$$

is finite.

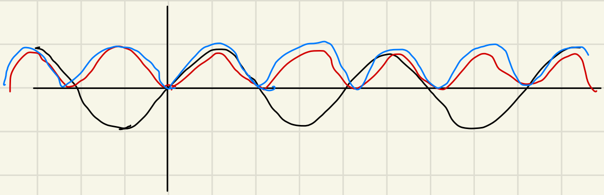
?

$0 \le \sin^2 x \le 1 \quad \forall x. \qquad 0 \le (\sin^2 x)^{1/3} \le 1 \quad \forall x.$

$\sin x$

$\sin^2 x$

$(\sin^2 x)^{1/3}$



$(\sin^2 x)^{1/3} \ge \sin^2 x \quad \forall x$ because $t^{1/3} \ge t$ for t on $[0,1]$ and $\sin^2 x$ on $[0,1]$ $\forall x$.

$$\Rightarrow \frac{1}{x^2 (\sin^2 x)^{1/3}} \le \frac{1}{x^2 \sin^2 x} = \left(\frac{1}{x \sin x}\right)^2$$

$$\lim_{x \to \infty} \frac{1}{x^2 (\sin^2 x)^{1/3}}$$

125 p. 3

3. A collection of functions $\{f_\alpha\}_{\alpha\in\mathcal{A}} \subset L^1(\mu)$ on the measure space $(X, \mathcal{M}, \mu)$ is said to be *uniformly integrable* if

$$\lim_{M\to\infty} \sup_{\alpha\in\mathcal{A}} \int_{\{x:|f_\alpha(x)|>M\}} |f_\alpha| = 0.$$

a. Prove that if $f \in L^1$ then $\{f\}$ is uniformly integrable.

b. Prove that if $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ and $\{f_\beta\}_{\beta\in\mathcal{B}}$ are two collections of uniformly integrable functions then $\{f_\gamma\}_{\gamma\in\mathcal{A}\cup\mathcal{B}}$ is uniformly integrable.

c. Show that if $\mu(X) < \infty$ and $\{f_\alpha\}_{\alpha\in\mathcal{A}} \subset L^1(\mu)$ is uniformly integrable then

$$\sup_{\alpha\in\mathcal{A}} \int |f| d\mu < \infty.$$

Give an example to show that the conclusion fails without the condition $\mu(X) < \infty$.

d. Again let $\mu(X) < \infty$ and suppose $\{f_n\}_{n=0}^\infty \subset L^1(\mu)$ such that $f_n \to f_0$ a.e. and $\int |f_n| d\mu \to \int |f_0| d\mu$. Prove that $\{f_n\}_{n=0}^\infty$ is uniformly integrable. Hint: Consider some ϕ_M, a continuous, bounded function on $[0,\infty)$, equal to 0 on $[M,\infty)$, for which $|t|\mathbf{1}\{|t| > M\} \le |t| - \phi_M(|t|)$.

a) NTS $\lim_{M\to\infty} \int_{\{x:|f|>M\}} |f| = 0$ for $f \in L^1$.

We have for any $M \in \mathbb{N}$:

$$\int |f| = \sum_{k=1}^{\infty} \int_{\{x:\, k-1<|f|\le k\}} |f|$$
$$= \sum_{k=1}^{M} \int_{\{x:\, k-1<|f|\le k\}} |f| + \int_{\{x:|f|>M\}} |f|$$

$$\Rightarrow \lim_{M\to\infty} \int_{\{x:|f|>M\}} |f| = \lim_{M\to\infty}\left(\int|f| - \sum_{k=1}^{M}\int_{\{x:\, k-1<|f|\le k\}}|f|\right)$$
$$= \int|f| - \sum_{k=1}^{\infty}\int_{\{x:\, k-1<|f|\le k\}}|f|$$
$$= \int|f| - \int|f|$$
$$= 0.$$

$\square$

3. A collection of functions $\{f_\alpha\}_{\alpha\in\mathcal{A}} \subset L^1(\mu)$ on the measure space $(X, \mathcal{M}, \mu)$ is said to be *uniformly integrable* if

$$\lim_{M\to\infty} \sup_{\alpha\in\mathcal{A}} \int_{\{x:|f_\alpha(x)|>M\}} |f_\alpha| = 0.$$

b. Prove that if $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ and $\{f_\beta\}_{\beta\in\mathcal{B}}$ are two uniformly integrable collections of ~~uniformly integrable~~ functions then $\{f_\gamma\}_{\gamma\in\mathcal{A}\cup\mathcal{B}}$ is uniformly integrable.

b)
$$\lim_{M\to\infty} \sup_{\gamma\in A\cup B} \int_{\{x:|f_\gamma(x)|>M\}} |f_\gamma|$$
$$\leq \lim_{M\to\infty} \max\left\{ \sup_{\alpha\in A} \int_{\{x:|f_\alpha(x)|>M\}} |f_\alpha| ,\ \sup_{\beta\in B} \int_{\{x:|f_\beta(x)|>M\}} |f_\beta| \right\}$$
$$\leq \lim_{M\to\infty} \sup_{\alpha\in A} \int_{\{x:|f_\alpha(x)|>M\}} |f_\alpha| + \sup_{\beta\in B} \int_{\{x:|f_\beta(x)|>M\}} |f_\beta|$$
$$= 0 + 0 = 0.$$

Thus $\lim_{M\to\infty} \sup_{\gamma\in A\cup B} \int_{\{x:|f_\gamma(x)|>M\}} |f_\gamma| = 0.$ $\square$

3. A collection of functions $\{f_\alpha\}_{\alpha\in\mathcal{A}} \subset L^1(\mu)$ on the measure space $(X, \mathcal{M}, \mu)$ is said to be *uniformly integrable* if

$$\lim_{M\to\infty} \sup_{\alpha\in\mathcal{A}} \int_{\{x:|f_\alpha(x)|>M\}} |f_\alpha| = 0.$$

c. Show that if $\mu(X) < \infty$ and $\{f_\alpha\}_{\alpha\in\mathcal{A}} \subset L^1(\mu)$ is uniformly integrable then

$$\sup_{\alpha\in\mathcal{A}} \int |\underset{f_\alpha}{f}| d\mu < \infty.$$

Give an example to show that the conclusion fails without the condition $\mu(X) < \infty$.

c) $$\int |f_\alpha|\, d\mu = \int_{\{x: |f_\alpha| \le M\}} |f_\alpha|\, d\mu + \int_{\{x: |f_\alpha| > M\}} |f_\alpha|\, d\mu$$

for any $M \in \mathbb{N}$. Thus, we have $\forall M$:

$$\begin{aligned}
\sup_\alpha \int |f_\alpha|\, d\mu &= \sup_\alpha \left(\int_{\{x: |f_\alpha| \le M\}} |f_\alpha|\, d\mu + \int_{\{x: |f_\alpha| > M\}} |f_\alpha|\, d\mu \right) \\
&\le \sup_\alpha \int_{\{x: |f_\alpha| \le M\}} |f_\alpha|\, d\mu + \sup_\alpha \int_{\{x: |f_\alpha| > M\}} |f_\alpha|\, d\mu \\
&\le \mu(X) M + \underbrace{\sup_\alpha \int_{\{x: |f_\alpha| > M\}} |f_\alpha|\, d\mu}
\end{aligned}$$

$\{f_\alpha\}$ uniformly integrable $\Rightarrow$ $\exists$ M s.t. this is finite.

Hence $\sup_\alpha \int |f_\alpha|\, d\mu$ is finite.

$\square$

Let $f_n = \chi_{[n, 2n]}$

$$\lim_{M\to\infty} \sup_n \int_{\{x: |f_n(x)| > M\}} |f_n| = 0$$ since

$$\sup_n \int_{\{x: |f_n(x)| > M\}} |f_n| = 0 \quad \forall\ M \ge 1.$$

But $$\sup_n \int |f_n|\, d\mu = \sup_n n = \infty.$$

3. A collection of functions $\{f_\alpha\}_{\alpha\in\mathcal{A}} \subset L^1(\mu)$ on the measure space $(X,\mathcal{M},\mu)$ is said to be *uniformly integrable* if

$$\lim_{M\to\infty}\sup_{\alpha\in\mathcal{A}}\int_{\{x:|f_\alpha(x)|>M\}}|f_\alpha| = 0.$$

d. Again let $\mu(X) < \infty$ and suppose $\{f_n\}_{n=0}^\infty \subset L^1(\mu)$ such that $f_n \to f_0$ a.e. and $\int|f_n|d\mu \to \int|f_0|d\mu$. Prove that $\{f_n\}_{n=0}^\infty$ is uniformly integrable. Hint: Consider some ϕ_M, a continuous, bounded function on $[0,\infty)$, equal to 0 on $[M,\infty)$, for which $|t|\mathbf{1}\{|t|>M\} \le |t| - \phi_M(|t|)$.

d)

$$\int_0^1 x^{-1/2} = 2x^{1/2}\Big]_0^1 = 2$$

Must $\int|f_0|d\mu < \infty$?

$$\int_0^1 x^{-n/n+1} = (n+1)x^{1/n+1}\Big]_0^1 = n+1$$

$$f_n(x) = x^{-n/n+1}$$

ϕ_M

$$f_n(x) \to \frac{1}{x} = f(x) \text{ a.e. } \checkmark$$

$$\int_0^1 |f_n| \to \int_0^1 |f|$$
$$= n+1 \to = \infty \quad \text{as } n\to\infty, \checkmark$$

$$x^{-n/n+1} = M$$
$$x = M^{-n+1/n}$$

$$\lim_{M\to\infty}\underbrace{\sup_n \int_{\{x:|f_n(x)|>M\}} |f_n(x)|} =$$

$$= \sup_n \int_{\{x:|f_n(x)|>M\}} x^{-n/n+1} = \sup_n \int_0^{M^{-n+1/n}} x^{-n/n+1}$$

$$= \sup_n (n+1)x^{1/n+1}\Big]_0^{M^{-n+1/n}}$$

$$= \sup_n (n+1) M^{-1/n}$$

$$= \infty$$

$$\lim_{M\to\infty}\sup_{\alpha\in A}\int \phi_M(f_n) \to \int \phi_M(f_0)$$

b/c bounded ϕ_M bounded

DCT

3. A collection of functions $\{f_\alpha\}_{\alpha \in \mathcal{A}} \subset L^1(\mu)$ on the measure space $(X, \mathcal{M}, \mu)$ is said to be *uniformly integrable* if

$$\lim_{M\to\infty} \sup_{\alpha\in\mathcal{A}} \int_{\{x:|f_\alpha(x)|>M\}} |f_\alpha| = 0.$$

d. Again let $\mu(X) < \infty$ and suppose $\{f_n\}_{n=0}^\infty \subset L^1(\mu)$ such that $f_n \to f_0$ a.e. and $\int |f_n| d\mu \to \int |f_0| d\mu$. Prove that $\{f_n\}_{n=0}^\infty$ is uniformly integrable. Hint: Consider some ϕ_M, a continuous, bounded function on $[0, \infty)$, equal to 0 on $[M, \infty)$, for which $|t|\mathbf{1}\{|t| > M\} \le |t| - \phi_M(|t|)$.

$$\phi_M(|t|) \le |t|(1 - \mathbb{1}_{\{|t|>M\}})$$
$$= |t|\,\mathbb{1}_{\{|t|\le M\}}$$

d) First, assume $f_0 \in L^1(\mu)$.

$\exists\ N$ s.t. $\forall\ n \ge N$, $\int |f_n|\, d\mu < \int |f_0|\, d\mu + \varepsilon$

By parts (a) and (b), using induction, we see that $\{f_n\}_{n=0}^N$ is uniformly integrable.

125p.4

4. Let $\mathbb{M}$ be the collection of all finite measures on the measure space $(X, \mathcal{M})$.

a. Show that

$$d(\nu, \lambda) = 2 \sup_{E \in \mathcal{M}} |\nu(E) - \lambda(E)|$$

defines a metric on $\mathbb{M}$.

b. For any $\mu \in \mathbb{M}$ that dominates measures ν and λ in $\mathbb{M}$ with $\nu(X) = \lambda(X) = 1$, let

$$p = \frac{d\nu}{d\mu} \quad \text{and} \quad q = \frac{d\lambda}{d\mu}.$$

Prove

$$d(\nu, \lambda) = \int |p(x) - q(x)| \, d\mu = 2\left(1 - \int (\min\{p(x), q(x)\}) \, d\mu\right).$$

Hint: notice that $\mu(E) - \lambda(E) = \lambda(E^c) - \nu(E^c)$.

a) • $d(\nu,\lambda) \geq 0$ ✓

• $\nu = \lambda \Rightarrow \quad d(\nu,\lambda) = 2 \sup |\nu(E) - \lambda(E)| = 2 \sup \{0\} = 0$. ✓

$d(\nu,\lambda) = 0 \Rightarrow \quad 2 \sup_E |\nu(E) - \lambda(E)| = 0$

$\Rightarrow \quad \nu(E) = \lambda(E) \quad \forall \ E \in \mathcal{M}$ ✓

• $d(\nu,\lambda) = d(\lambda,\nu)$ since $|\nu(E) - \lambda(E)| = |\lambda(E) - \nu(E)|$. ✓

• NTS $d(\nu,\mu) \leq d(\nu,\lambda) + d(\lambda,\mu)$

$2 \sup |\nu(E) - \mu(E)| \leq 2 \sup |\nu(E) - \lambda(E)| + 2 \sup |\lambda(E) - \mu(E)|$?

Yes, b/c $|\nu(E) - \mu(E)| \leq |\nu(E) - \lambda(E)| + |\lambda(E) - \mu(E)|$ ✓

$\square$

b. For any $\mu \in \mathbb{M}$ that dominates measures ν and λ in $\mathbb{M}$ with $\nu(X) = \lambda(X) = 1$, let

$$p = \frac{d\nu}{d\mu} \quad \text{and} \quad q = \frac{d\lambda}{d\mu}.$$

Prove

$$d(\nu, \lambda) = \int |p(x) - q(x)| \, d\mu = 2\left(1 - \int (\min\{p(x), q(x)\}) \, d\mu\right).$$

Hint: notice that $\mu(E) - \lambda(E) = \lambda(E^c) - \nu(E^c)$.

b)