

Real, Spring 2012

(1) f & g real-valued & integrable on
a σ -finite space X .

$$\text{Set } F_t = \{x \in X : f(x) > t\} \quad \& \\ G_t = \{x \in X : g(x) > t\}.$$

$$\text{Show } \int_X |f - g| d\mu = \int_{\mathbb{R}} \mu((F_t - G_t) \cup (G_t - F_t)) dt.$$

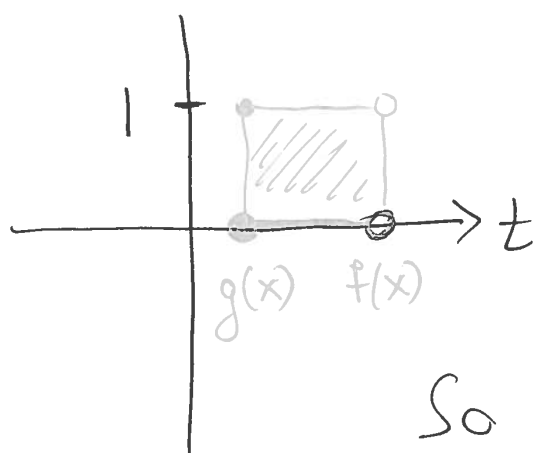
$$\text{Observe } \mu((F_t - G_t) \cup (G_t - F_t))$$

$$= \int_{\mathbb{R}} \chi_{F_t - G_t}^{(x)} d\mu + \int_{\mathbb{R}} \chi_{G_t - F_t}^{(x)} d\mu. \quad \text{By Tonelli,}$$

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \chi_{F_t - G_t}^{(x)} + \chi_{G_t - F_t}^{(x)} d\mu dt =$$

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \chi_{F_t - G_t}(x) dt d\mu + \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_{G_t - F_t}(x) dt d\mu.$$

Observe $\chi_{F_t - G_t}(x) \stackrel{\text{think fixed } x}{=} \begin{cases} 1, & x \in F_t - G_t \\ 0, & \text{otherwise} \end{cases}$



$$= \begin{cases} 1, & f(x) > t \text{ \& } g(x) \leq t \\ 0, & \text{otherwise} \end{cases}$$

So $\int_{\mathbb{R}} \chi_{F_t - G_t}(x) dt = \begin{cases} f(x) - g(x) \\ \text{on } \{x: f(x) \geq g(x)\} \end{cases}$

Hence, in summary, $\int_{\mathbb{R}} \mu(G_t \Delta F_t) dt = \begin{cases} \text{and} \\ 0 \text{ elsewhere.} \end{cases}$

$$= \int_{\{x: f(x) > g(x)\}} f(x) - g(x) dt + \int_{\{x: g(x) > f(x)\}} g(x) - f(x) dt = \int_{\mathbb{R}} |f - g| dt.$$

$$\{x: f(x) > g(x)\} \quad \{x: g(x) > f(x)\} \quad \mathbb{R}$$



(2) Show $\int_{\pi}^{\infty} \frac{dx}{x^2 (\sin^2 x)^{\frac{1}{3}}}$ is finite.

The singularities occur at $x = n\pi$, $n \geq 1$.

We aim to show $\sum_{n=1}^{\infty} \int_{n\pi-c}^{n\pi+c} \frac{dx}{x^2 \sin^{\frac{2}{3}}(x)} < \infty$

for c sufficiently small.

Let $1 < \alpha < \frac{3}{2}$. Then for $c \ll 1$,

$$|x - n\pi| < c \Rightarrow |\sin x| \geq |x - n\pi|^{\alpha}$$

$$\text{So } \int_{n\pi-c}^{n\pi+c} \frac{dx}{x^2 \sin^{\frac{2}{3}}(x)} \leq \frac{1}{(n\pi-c)^2} \int_{n\pi-c}^{n\pi+c} \frac{dx}{\sin^{\frac{2}{3}}(x)}$$

$$\leq \frac{1}{(n\pi-c)^2} \int_{n\pi-c}^{n\pi+c} \frac{dx}{|x - n\pi|^{\frac{2}{3}\alpha}}$$

Since $1 - \frac{2}{3}\alpha > 0$, we have

$$\int_{n\pi-c}^{n\pi+c} \frac{dx}{(x - n\pi)^{\frac{2}{3}\alpha}} = 2 \int_0^c \frac{dx}{x^{\frac{2}{3}\alpha}} = \frac{2}{1 - \frac{2}{3}\alpha} (c^{1 - \frac{2}{3}\alpha} - 0).$$

we gather

$$\sum_{n=1}^{\infty} \int_{n\pi-c}^{(n+1)\pi+c} \frac{dx}{x^2 \sin^{2/3}(x)} \leq \frac{2C}{1-\frac{2}{3}d} \sum_{n=1}^{\infty} \frac{1}{(n\pi-c)^2} < \infty.$$

It remains to show
$$\sum_{n=1}^{\infty} \int_{n\pi+c}^{(n+1)\pi-c} \frac{dx}{x^2 \sin^{2/3}(x)} < \infty.$$

Let M be s.t. $\sin x \geq M$ on $[c, \pi-c]$.

Then

$$\sum_{n=1}^{\infty} \int_{n\pi+c}^{(n+1)\pi-c} \frac{dx}{x^2 \sin^{2/3}(x)} \leq \frac{1}{M^{2/3}} \int_{\pi+c}^{\infty} \frac{dx}{x^2} < \infty.$$

We conclude
$$\int_{\pi}^{\infty} \frac{dx}{x^2 \sin^{2/3}(x)} < \infty. \quad \square$$

(3) $\{f_n\} \subset L^1(X)$ are

uniformly integrable if

$$\lim_{n \rightarrow \infty} \sup_{\substack{A \\ \{x: |f_n(x)| > n\}}} \int_A |f_n| = 0.$$

(a) If $f \in L^1$, then $\{f\}$ is uniformly integrable.

$$\int |f| = \sum_{n=0}^{\infty} \int |f| \chi_{\{x: n \leq |f(x)| < n+1\}} < \infty$$

$$\Rightarrow \text{tail} \sum_{n=M}^{\infty} \int |f| \chi_{\{x: n \leq |f(x)| < n+1\}}$$

$$= \int |f| \chi_{\{x: |f(x)| \geq M\}} \rightarrow 0 \text{ as } M \rightarrow \infty.$$

□

(b) If $\{f_\alpha\}_{\alpha \in A}$ and $\{f_\beta\}_{\beta \in B}$, then

$\{f_\gamma\}_{\gamma \in A \cup B}$ is uniformly integrable.

For a fixed M ,

$$\sup_{\gamma \in A \cup B} \int |f_\gamma| \chi_{\{x: |f_\gamma(x)| > M\}} \leq \max \left\{ \sup_{\alpha \in A} \int |f_\alpha| \chi_{\{x: |f_\alpha(x)| > M\}}, \sup_{\beta \in B} \int |f_\beta| \chi_{\{x: |f_\beta(x)| > M\}} \right\}$$

$\longrightarrow \bigcirc$ as $M \longrightarrow \infty$. $\square$

(c) If $\mu(X) < \infty$ and $\{f_k\}$ is uniformly integrable, then

$$\sup_k \int |f_k| < \infty.$$

This is not necessarily true if $\mu(X) = \infty$.

$$\text{Know } \lim_{M \rightarrow 0} \sup_k \int_{\{x: |f_k(x)| > M\}} |f_k| = 0.$$

Fix $\varepsilon > 0$. Choose M so large s.t.

$$\int_{\{x: |f_k(x)| > M\}} |f_k| \leq \sup_k \int_{\{x: |f_k(x)| > M\}} |f_k| < \varepsilon.$$

$$\text{Also, } \sup_k \int_{\{x: |f_k(x)| \leq M\}} |f_k| \leq M \mu(X).$$

(Combining these, $\sup_k \int |f_k| < \infty$.)

Observe $f_n = \begin{cases} 1 & \text{on } [-n, 0] \\ 0 & \text{elsewhere} \end{cases}$ on $\mathbb{R}$

is in L^1 , $\sup_n \int_{\{x: |f_n(x)| > 1\}} |f_n| = \sup_n 0 = 0$.

And $\sup_n \int |f_n| = \infty$. $\square$

(d) $\mu(X) < \infty$. $\{f_n\} \subset L^1(\mu)$ s.t.

$$f_n \rightarrow f_0 \text{ a.e. and } \int |f_n| \rightarrow \int |f_0|$$

Show $\{f_n\}$ is uniformly integrable.

NOTE! we must assume $\int |f_0| < \infty$.

Indeed, if $\int |f_0| = \infty$, then, assuming $\{f_n\}$ is uniformly integrable,

$$\int |f_n| \leq M\mu(X) + \sup_n \int_{\{x: |f_n(x)| > M\}} |f_n|$$

$$\rightarrow \leq M\mu(X) + \varepsilon \text{ for } M \gg 1.$$

Hence, $\sup_n \int |f_n| < \infty$, violating $\int |f_n| \rightarrow \infty$.

~~So we must assume $\int |f_0| < \infty$.~~
~~HOWEVER! Assuming this, we will~~
~~prove the result WITHOUT~~
~~assuming $\int |f_0| < \infty$.~~

So assume $\int |f_0| < \infty$. Fix $\varepsilon > 0$
 First we show Choose M s.t. $\int |f_0| < \varepsilon$, for all $M > M'$
 $\{x: |f_0(x)| > M\}$

$$\lim_{n \rightarrow \infty} \int_{\{x: |f_n(x)| > M\}} |f_n| = \int_{\{x: |f_0(x)| > M\}} |f_0|$$

except possibly on a small set, where M is fixed.

Let $g_n = |f_n| \chi_{\{x: |f_n(x)| > M\}}$ and

$$g_0 = |f_0| \chi_{\{x: |f_0(x)| > M\}}.$$

$g_n \rightarrow g_0$ pointwise a.e., $\wedge$ Indeed, since $f_n(x) \rightarrow f_0(x)$
 a.e., if $|f_0(x)| > M$, then eventually $|f_n(x)| > M$, so $g_n(x) \rightarrow g_0(x)$ where
 $|f_0(x)| > M$. However, if $|f_0(x)| \leq M$, it is possible $\wedge$ that
 (when $|f_0(x)| = M$)

$|f_n(x)| > M$ for arbitrarily large n , in which case

$g_n(x) \not\rightarrow g_0(x)$. So we must exclude these points,

which is where the aforementioned 'small sets'

come in,

Define $E_M = \left\{ x : \begin{array}{l} |f_0(x)| = M \text{ and} \\ |f_n(x)| > M \text{ for} \\ \text{arbitrarily large } n \end{array} \right\}.$

~~Clearly the E_M are disjoint. Then since $\mu(X) < \infty$, there is M so large s.t. $\sum_{M' \geq M} \mu(E_{M'}) < \varepsilon$, where $\varepsilon > 0$ is fixed. Set $E = \bigcup_{M' \geq M} E_{M'}$.~~

Our argument from above shows

$$g_n \rightarrow g_0 \text{ a.e. on } X - E_M$$

On this set $X - E_M$, we then have $g_n \leq |f_n|$, $g_n \rightarrow g_0$ a.e.,
 $|f_n| \rightarrow |f_0|$ a.e.

$g_n, |f_n|, g_0, |f_0| \in L^1(X - E_M)$. We want to use the

generalized DCT, which means we also need

$$\int_{X - E_M} |f_n| \rightarrow \int_{X - E_M} |f_0| \quad (*)$$

Why is this true?

This also follows from the generalized DCT.

Since $f_n, f_0 \in L^1$ and $f_n \rightarrow f_0$ a.e. and $\int |f_n| \rightarrow \int |f_0|$ we have $|f_n - f_0| \rightarrow 0$, $|f_n - f_0| \leq \frac{|f_n| + |f_0|}{2} \in L^1$, $|f_n| \xrightarrow{+|f_0|} |f_0| + |f_0|$ a.e. and $\int |f_n| + |f_0| \rightarrow 2 \int |f_0|$. By the generalized DCT, $\int |f_n - f_0| \rightarrow 0$. Consequently, $\int_F |f_n - f_0| \rightarrow 0$, whence $\left| \int_F |f_n| - \int_F |f_0| \right| \leq \int |f_n - f_0| \rightarrow 0$, whence $\int_F |f_n| \rightarrow \int_F |f_0|$; take $F = X - E_M$. (*) follows.

By the generalized DCT again, we conclude

$$\lim_{n \rightarrow \infty} \int_{\{x: |f_n(x)| > M\} - E_M} |f_n| = \int_{\{x: |f_0(x)| > M\} - E_M} |f_0| \quad (\star)$$

(**) And on the bad set E_M , by above, i.e. $(F = E_M)$,

$$\lim_{n \rightarrow \infty} \int_{E_M} |f_n| = \int_{E_M} |f_0|.$$

Hence,

$$\int_{\{x: |f_n(x)| > M\} \cap E} |f_n| \leq \int_{E_M} |f_n| \leq \int_{E_M} |f_0| + \varepsilon \leq M \mu(E) + \varepsilon \leq M \varepsilon + \varepsilon \quad (n \text{ sufficiently large})$$

Observe now that

$$\int_{E_M} |f_0| \leq \int_{\{x: |f_0(x)|=M\}} |f_0| \leq \int_{\{x: |f_0(x)| \geq M\}} |f_0| \longrightarrow 0 \text{ as } M \rightarrow \infty.$$

Referring to our choice of M' at the beginning of the solution, we may assume ^{when} M is so large that

$$\int_{\{x: |f_0(x)| > M\}} |f_0| < \varepsilon \quad \& \quad \int_{E_M} |f_0| < \varepsilon.$$

Fix such an M .

Then by $(*)$ & $(**)$, we may take n sufficiently large s.t.

$$\begin{aligned} \int_{\{x: |f_n(x)| > M\} - E_M} |f_n| &\leq \int_{\{x: |f_0(x)| > M\} - E_M} |f_0| + \varepsilon \\ &\leq \int_{\{x: |f_0(x)| > M\}} |f_0| + \varepsilon \\ &\leq 2\varepsilon \end{aligned}$$

and $\int_{E_M} |f_n| \leq \int_{E_M} |f_0| + \varepsilon \leq 2\varepsilon$. Therefore,

$$\int_{\{x: |f_n(x)| > M\}} |f_n| = \int_{\{x: |f_n(x)| > M\} - E_M} |f_n| + \int_{\{x: |f_n(x)| > M\} \cap E_M} |f_n| \leq 2\varepsilon + \int_{E_M} |f_n| \leq 4\varepsilon.$$

This is true for a single value of M

but for n sufficiently large. Thus we have

proved
$$\sup_{n \geq N} \int_{\{x: |f_n(x)| > M\}} |f_n| \leq 4\varepsilon.$$

But
$$\sup_{n \geq N} \int_{\{x: |f_n(x)| > M\}} |f_n|$$
 decreases w/ M .

Consequently,
$$\lim_{M \rightarrow 0} \sup_{n \geq N} \int_{\{x: |f_n(x)| > M\}} |f_n| = 0.$$

By part (b), we may add in those f_n for $n < N$ for free. We conclude $\{f_n\}$ is uniformly integrable



(4) $\mathcal{M}$ collection of finite measures
on a measure space $(X, \mathcal{M})$.

(a) Show $d(\nu, \lambda) \equiv 2 \sup_{E \in \mathcal{M}} |\nu(E) - \lambda(E)|$

is a metric on $\mathcal{M}$.

$$2 \sup_{E \in \mathcal{M}} |\nu(E) - \lambda(E)| = 0 \Rightarrow \nu(E) = \lambda(E) \quad \forall E \in \mathcal{M} \\ \Rightarrow \nu = \lambda.$$

$d(\nu, \lambda) = d(\lambda, \nu)$ by definition.

$$|\nu(E) - \lambda(E)| \leq |\nu(E) - \mu(E)| + |\mu(E) - \lambda(E)| \\ \leq \sup_{F \in \mathcal{M}} |\nu(F) - \mu(F)| + \sup_{F \in \mathcal{M}} |\mu(F) - \lambda(F)|$$

$$\Rightarrow d(\nu, \lambda) \leq d(\nu, \mu) + d(\mu, \lambda). \quad \square$$

For any finite μ s.t. $\nu, \lambda \ll \mu$, set
 $\frac{d\nu}{d\mu}$ and $\frac{d\lambda}{d\mu}$. Prove
 $\frac{d\nu}{d\mu} \geq 0$ and $\frac{d\lambda}{d\mu} \geq 0$

$$d(\nu, \lambda) = \int |p - q| = 2 \left(1 - \int \min(p, q) \right).$$

Hint: $v(E) - \lambda(E) = \lambda(E^c) - v(E^c)$

Since $v(E) + v(E^c) = 1 = \lambda(E) + \lambda(E^c)$.

Observe $\int |p - q| = \int p - q + \int q - p$
 $E = \{x: p(x) \geq q(x)\} \quad E^c = \{x: q(x) > p(x)\}$
 $= v(E) - \lambda(E) + \lambda(E^c) - v(E^c)$

And $2(1 - \int \min(p, q)) = 2 - 2\left(\int_{E^c} p + \int_E q\right)$

$= 2 - 2v(E^c) - 2\lambda(E)$

$\stackrel{?}{=} v(E) - \lambda(E) + \lambda(E^c) - v(E^c)$

$\Leftrightarrow 2 = v(E) + \lambda(E) + \lambda(E^c) + v(E^c) = 1 + 1 \quad \checkmark$

For any F , we have $2|v(F) - \lambda(F)| \stackrel{\text{Hint}}{=}$

$|v(F) - \lambda(F)| + |\lambda(F^c) - v(F^c)| = \left| \int_F p - q \right| + \left| \int_{F^c} q - p \right|$
 $= \left| \int_{E \cap F} p - q - \int_{E^c \cap F} q - p \right| + \left| \int_{E^c \cap F^c} q - p - \int_{E \cap F^c} p - q \right| \leq \int_{E \cap F} p - q + \int_{E^c \cap F^c} q - p$
 $\leq \int_E p - q + \int_{E^c} q - p = \int |p - q|. \quad \therefore d(v, \lambda) \leq \int |p - q|.$

... And $\int |p-q|$ is achieved:

Taking $F = E$ as previously, it is clear
that the two inequalities become equalities,

