Spring '11:

1. $A \subseteq \mathbb{R}$ and for each $\varepsilon > 0$, $\exists$ Leb-meas. E, F s.t. $E \subseteq A \subseteq F$ and $m(F \setminus E) < \varepsilon$. Show A is Leb-measurable

- For each $\varepsilon > 0$, $\exists$ Leb-meas E, F s.t. $E \subseteq A \subseteq F$ and $m(F \setminus E) < \varepsilon$, i.e. in particular, $\exists F_n, E_n$ s.t. $E_n \subseteq A \subseteq F_n$ and $m(F_n \setminus E_n) < \frac{1}{n}$ for each n
- Now let $E = \bigcup E_n$, $F = \bigcap F_n$, hence $E \subseteq A \subseteq F$, and particularly, $F \setminus E \subseteq F_n \setminus E_n$ for each n, hence:

$$m(F \setminus E) \leq \lim_{n \to \infty} m(F_n \setminus E_n) < \lim_{n \to \infty} \frac{1}{n} = 0$$

$\Rightarrow m(F \setminus E) = 0.$

- Since Lebesgue meas is complete every subset of a null set is measurable (and meas. 0)
In particular $A \setminus E \subseteq F \setminus E$, hence $m(A \setminus E) = 0$,
here $A = E \cup N$ where N is Lebesgue null, hence A is union of two measurable sets, hence measurable.

2. $f > 0$ Leb-int on $[0,1]$. Show: $\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^1 (f^\varepsilon - 1)\, dm = \int_0^1 \log f \, dm$

- First see that $\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^1 (f^\varepsilon - 1)\, dm = \lim_{k \to 0} \frac{1}{1/k} \int_0^1 (f^{1/k} - 1)\, dm$.

Now consider the sequence $g_n = \frac{f^{1/n} - 1}{1/n} = n(f^{1/n} - 1)$.

<u>case $f > 1$</u>: this implies that $\log f > 0$; now consider

$$\frac{d}{dn} g_n = \frac{d}{dn}\left(n(f^{1/n} - 1)\right) = (f^{1/n} - 1) + n\left(-\frac{\log f}{n^2} e^{\frac{\log f}{n}}\right)$$

$$= f^{1/n}\left(1 - \frac{\log f}{n}\right) - 1$$

$$\leq \left(1 - \frac{\log f}{n}\right) - 1 = \frac{-\log f}{n} < 0$$

hence $g_n(x)$ is decreasing as $n \to \infty$ for all x.
Since f integrable, $\int g_1 = \int (f - 1) < \infty$, so may apply DCT
since now $|g_n| \leq |g_1| \in L^1$ for all n (since g_n decreasing) ?

case $f<1$: this implies that $\log f < 0$, so:

$$\frac{d}{dn} g_n = \underbrace{f^{1/n}}_{\leq 1}\Big(\underbrace{1-\frac{\log f}{n}}_{\geq 1}\Big) - 1 \geq \Big(1-\frac{\log f}{n}\Big) - 1 = \frac{-\log f}{n} > 0,$$

here $g_n(x)$ is increasing for all x, hence we may apply MCT.

- Apply MCT: $\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\int_0^1 (f^{\varepsilon}-1)\,dm = \int_0^1 \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}(f^{\varepsilon}-1)\,dm$

$$= \int_0^1 \frac{d}{d\varepsilon}(f^{\varepsilon})\Big|_{\varepsilon=0}\,dm = \int_0^1 \log f\, e^{\varepsilon \log f}\Big|_{\varepsilon=0}\,dm = \int_0^1 \log f\,dm$$

(3.) $f \in L^1(\mathbb{R})$ abs cont and $\lim_{h\downarrow 0} \int_{\mathbb{R}} \left|\frac{f(x+h)-f(x)}{h}\right| dx = 0$. Show $f=0$ a.e.

Recall $\lim_{h\downarrow 0} \frac{f(x+h)-f(x)}{h} = \lim_{n\to\infty} \frac{f(x+\frac{1}{n})-f(x)}{(\frac{1}{n})}$

Consider now the sequence $g_n(x) = \frac{f(x+\frac{1}{n})-f(x)}{(\frac{1}{n})}$ and:

$$\int |f'(x)|\,dm = \int \liminf_{n\to\infty} \frac{f(x+\frac{1}{n})-f(x)}{(\frac{1}{n})}$$

exists by abs. cont. of f

$$= \int \liminf_{n\to\infty} |g_n| \leq \liminf_{n\to\infty} \int |g_n|\,dm \quad \text{(FATOU)}$$

$$= \liminf_{n\to\infty} \int \left| \frac{f(x+\frac{1}{n})-f(x)}{\frac{1}{n}} \right| dm$$

$$= \liminf_{h\to 0} \int \left| \frac{f(x+h)-f(x)}{h} \right| dm = 0$$

hypothesis

Hence: $\int |f'(x)|\,dm = 0 \Rightarrow |f'(x)| = 0$ a.e.

$\Rightarrow f'(x) = 0$ a.e.

$\Rightarrow f(x) = \text{const. a.e. } x$

$\Rightarrow f(x) \equiv 0$ a.e. x since f is L^1.

(4.) (X, μ) meas. space, $\mu(X) = 1$, $F_1, \dots, F_7$ measurable with $\mu(F_j) \geq \frac{1}{2}$ for all $j = 1, \dots, 7$.

(a) Show $\exists\, i_1 < i_2 < i_3 < i_4$ s.t. $F_{i_1} \cap F_{i_2} \cap F_{i_3} \cap F_{i_4} \neq \emptyset$.

- Suppose $\nexists$ non-empty 4-way intersection, i.e. $\sum_{i=1}^{7} \chi_{F_i}(x) < 4$ for all $x \in X$ (i.e. $\sum_{i=1}^{7} \chi_{F_i}(x) \leq 3$)

- Now consider the integral:

$$3 \geq \int_X \sum_{i=1}^{7} \chi_{F_i}(x)\, d\mu = \sum_{i=1}^{7} \int \chi_{F_i}(x)\, d\mu = \sum_{i=1}^{7} \mu(F_i)$$

since $\mu(X) = 1$

$$\geq \sum_{i=1}^{7} \frac{1}{2} = \frac{7}{2}$$

hence $3.5 \leq \int_X \sum_{i=1}^{7} \chi_{F_i}(x)\, d\mu \leq 3$, a contradiction

Therefore a nonempty 4-way intersection must exist.

(b) $f_n \in L^1(m)$ with $f_n \geq 0$, $\int_0^{1/n} f_n\, dm \geq \frac{1}{2}$ $\forall n \geq 1$.

Show that $\int_0^1 \sup_n f_n(x)\, dm = \infty$

- Define the measure $\nu(E) = \int_E \sup_n f_n(x)\, dm$; by definition $\nu \ll m$, hence (Thm 3.5) for $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. $|m(E)| < \delta \Rightarrow |\nu(E)| < \varepsilon$

- Suppose that $\int_0^1 \sup_n f_n(x)\, dm = \nu([0,1]) < \infty$.

Choose $\varepsilon = \frac{1}{2}$, hence $\exists\, \delta > 0$ s.t. $m(A) < \delta \Rightarrow \nu(A) < \frac{1}{2}$;

now choose m s.t. $\frac{1}{m} < \delta$. Then, for $A = [0, \frac{1}{m}]$, we have:

$$m(A) < \delta \Rightarrow \nu(A) < \frac{1}{2} \Rightarrow \int_0^{1/m} \sup_n f_n(x)\, dm < \frac{1}{2},$$

but $\int_0^{1/m} \sup_n f_n(x)\, dm \geq \int_0^{1/m} f_m(x)\, dm \geq \frac{1}{2}$, a contradiction.