Real, Spring 2011

(1) $A \subset \mathbb{R}$. For each $\varepsilon > 0$ there exist E, F w/ $E \subset A \subset F$ and $m(F - E) < \varepsilon$. Show A is measurable.

Choose E_n, F_n corresponding to $\frac{1}{n}$.

Set $E = \bigcup E_n$ and $F = \bigcap F_n$. Then $E \subset A \subset F$. Write $A = E \cup (A - E)$. E is measurable. NTS $A - E \subset \overset{\text{measurable}}{N}$, $m(N) = 0$, so that by completeness A is measurable. Set $N = F - E \supseteq A - E$. For each n, $m(F_n - E_n) < \frac{1}{n}$. Since $F_n - \bigcup_m E_m \subset F_n - E_n$,

$$m\left(F_n - \bigcup_m E_m\right) \leq m(F_n - E_n) < \frac{1}{n}.$$

Since $\bigcap_m F_m - \bigcup_m E_m \subset F_n - \bigcup_m E_m$,

$$m(F - E) = m\left(\bigcap_m F_m - \bigcup_m E_m\right) \leq m\left(F_n - \bigcup_m E_m\right) < \frac{1}{n}.$$

Since n was arbitrary, $m(F - E) = m(N) = 0$, and we are done. $\square$

(2) $f > 0$, $f \in L^1([0,1])$.

Show $$\lim_{\varepsilon \downarrow 0} \int_0^1 \frac{f(x)^{\varepsilon} - 1}{\varepsilon}\, dx = \int_0^1 \log f(x)\, dx$$

First, $$\lim_{\varepsilon \downarrow 0} \frac{f^{\varepsilon}(x) - 1}{\varepsilon} = \lim_{\varepsilon \downarrow 0} \frac{f^{\varepsilon}(x) - f^{(0)}(x)}{\varepsilon - 0}$$

$$= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f(x)^{\varepsilon} = \log f(x)\, f^{(0)}(x) = \log f(x).$$

Observe $$\pm \frac{d}{d\varepsilon} \frac{f(x)^{\varepsilon} - 1}{\varepsilon} = \pm \frac{\log f(x) f(x)^{\varepsilon} \cdot 1 - f(x)^{\varepsilon} \cdot 1}{\varepsilon^2}$$

$$= \pm \frac{f(x)^{\varepsilon}(\log f(x) - 1)}{\varepsilon^2}$$

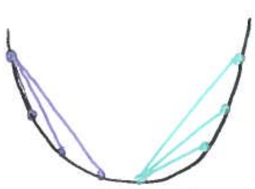
$$= \pm \frac{\log f(x) f(x)^{\varepsilon} \varepsilon - (f(x)^{\varepsilon} - 1)}{\varepsilon^2}$$

$$= \pm \frac{f(x)^{\varepsilon}(\log f(x)\varepsilon - 1) + 1}{\varepsilon^2}$$

$$\left.\frac{d^2}{d\varepsilon^2}\right|_{\varepsilon=0} f(x)^{\varepsilon} = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \log f(x)\, f(x)^{\varepsilon} = \log^2 f(x).$$

~~First, consider $\{x : |f(x)| \leq 1\}$, let x be in this set.~~

Since the second derivative at 0 $\log^2 f(x)$ is positive (or zero) the slopes of the secant lines $\frac{f^{\varepsilon}(x)-1}{\varepsilon}$ increase w/ ε for $\varepsilon \ll 1$. Moreover, the second derivative $\log^2 f(x) \cdot f(x)^{\varepsilon}$ is always nonnegative, so that $\frac{f^{\varepsilon}(x)-1}{\varepsilon}$ increases for all $\varepsilon > 0$. Hence, the sequence $\left\{\frac{f^{\varepsilon}-1}{\varepsilon}\right\}$ decreases on $[0,1]$ as $\varepsilon \downarrow 0$.

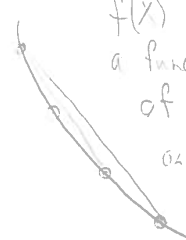


Consider $\{x : f(x) < 1\}$.

On this set $\log f(x) < 0$, so the first derivative at zero, $\log f(x)$, is negative; hence $\left\{\frac{f^{\frac{1}{n}}-1}{\frac{1}{n}}\right\}$ is a decreasing set of negative numbers, negative at least for sufficiently large n, x being fixed.

But, in fact, X need not be fixed; the first derivative $\log f(X) \cdot f^{\varepsilon}(X)$ is strictly negative (hence the slopes of the secant lines are always negative) for all X s.t. $0 < f(X) < 1$ and $\varepsilon > 0$ we gather $\left\{ \frac{f^{\frac{1}{n}} - 1}{\frac{1}{n}} \right\}$ is a decreasing sequence of negative functions. Hence we may apply the MCT to $\left\{ \frac{1 - f^{\frac{1}{n}}}{\frac{1}{n}} \right\}$, yielding

$$\lim_{n \to \infty} \int_{\{X : f(X) < 1\}} \frac{1 - f(x)^{\frac{1}{n}}}{\frac{1}{n}} = - \int_{\{X : f(X) < 1\}} \log f(x),$$



as desired, at least on this set.

On $\{X : f(X) \geq 1\}$ the first derivative $\log f(X) \cdot f^{\varepsilon}(X)$ is nonnegative for all $\varepsilon > 0$. So $\left\{ \frac{f^{\frac{1}{n}} - 1}{\frac{1}{n}} \right\}$ is nonnegative and decreasing. Since

(in its decreasing form)

note here we are using $\mu([0,1]) < \infty$!

$f - 1 = \frac{f^{\frac{1}{1}} - 1}{\frac{1}{1}} \in L^1([0,1])$, we may apply the MCT, yielding

$$\lim_{n \to \infty} \int_{\{X : f(X) \geq 1\}} \frac{f(x)^{\frac{1}{n}} - 1 \, dX}{\frac{1}{n}} = \int_{\{X : f(X) \geq 1\}} \log f(x) \, dX.$$

The same arguments apply to any decreasing sequence $x_n \downarrow 0$. We conclude

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int \frac{f^{\varepsilon} - 1}{\varepsilon} = \int \log f \, . \qquad \square$$

wrong! (see previous page)

~~(Note we did not require the finiteness of [0,1], so our argument is equally valid for all $f > 0$ in $L^1(\mathbb{R})$.)~~

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(3) Suppose $f \in L^1(\mathbb{R})$ is absolutely continuous and

$$\lim_{h \downarrow 0} \int_{\mathbb{R}} \left| \frac{f(x+h) - f(x)}{h} \right| dx = 0 .$$

Show $f = 0$ a.e.

(*) It suffices to show $f = \text{constant}$ a.e., for then $f \in L^1(\mathbb{R}) \Rightarrow f = 0$ a.e.

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*) Since $\displaystyle\int_{-n}^{n} \left| \frac{f(x+h) - f(x)}{h} \right| \leq \int_{\mathbb{R}} \left| \frac{f(x+h) - f(x)}{h} \right|$,

the hypothesis implies $\displaystyle\lim_{h \downarrow 0} \int_{-n}^{n} \left| \frac{f(x+h) - f(x)}{h} \right| = 0$.

Since $f = \text{constant}$ a.e. $\iff$ $f = \text{constant}$ a.e. on $[-n,n]$ $\forall n$ we have reduced to showing f is a constant on $[-n,n]$.

Since f is absolutely continuous on $[-n,n]$, its derivative exists a.e.

So $\lim\limits_{h\to 0}\left|\dfrac{f(x+h)-f(x)}{h}\right| = |f'(x)|$ converges for a.e. $x \in [-n,n]$.

Note that $f(x+h) \in L^1([-n,n])$ by change of variables ($f \in L^1([-n,n])$ and $x+h$ is increasing absolutely continuous). So $\left|\dfrac{f(x+h)-f(x)}{h}\right| \in L^1([-n,n])$ and the integrals make sense.

We forgot to mention $f' \in L^1([-n,n])$, specifically, $f(x) - f(-n) = \int_{-n}^{x} f'(t)\,dt$.

Fix $\varepsilon > 0$. Since m is finite on $[-n,n]$, by Egoroff's theorem $\left|\dfrac{f(x+h)-f(x)}{h}\right|$ converges uniformly on $[-n,n]$ to $|f'(x)|$ except possibly on a set of measure $< \varepsilon$.

So off of a small set, for h sufficiently small we may write $\left|\frac{f(x+h)-f(x)}{h}\right| \leq |f'(x)| + \varepsilon$.

Since $f' \in L^1$ and we are working on $[-n,n]$, $|f'(x)+\varepsilon|$ serves as a dominating function for $\left|\frac{f(x+h)-f(x)}{h}\right|$ for h sufficiently small and except on a set of measure $< \varepsilon$. Hence, on our uniformly convergent set we have by the DCT

$$\lim_{h \downarrow 0} \int \left|\frac{f(x+h)-f(x)}{h}\right| = \int |f'(x)| = 0$$

from which we conclude $f'(x) = 0$ off of a set of measure $< \varepsilon$. Since ε was arbitrary, $f'(x) = 0$ a.e. on $[-n,n]$. By the FTOC for Lebesgue theory, $f(x) - f(-n) = \int_{-n}^{x} f' = 0$ for a.e. x. Hence, f is actually a constant for a.e. $x \in [-n,n]$, and we conclude $f = 0$ a.e. on $\mathbb{R}$, as discussed at the beginning of the solution. $\square$

(4)(a) $E_i \subset [0,1]$, $i = 1, \dots, 7$ $\qquad m(E_i) \geq \frac{1}{2}$.

Show $\exists$ distinct $i_1, \dots, i_4$ s.t.

$$E_{i_1} \cap E_{i_2} \cap E_{i_3} \cap E_{i_4} \neq \emptyset.$$

Assume otherwise. Then

$$\sum_{i=1}^{7} \chi_{E_i} \leq 3 \quad \text{everywhere.}$$

Hence $3 = 3m([0,1]) \geq \int_{[0,1]} \sum_{i=1}^{7} \chi_{E_i} = \sum_{i=1}^{7} m(E_i) \geq 3.5$

a contradiction. $\square$

(b) $f_n \in L^1([0,1])$, $f_n \geq 0$ s.t.

$\int_{[0,\frac{1}{n}]} f_n \geq \frac{1}{2}$. Prove $\int_{[0,1]} \sup_n f_n(x) = \infty$.

$$\int_{[0,1]} \sup_n f_n(x) \underset{\text{positivity}}{=} \sum_{n=1}^{\infty} \int_{[\frac{1}{n+1}, \frac{1}{n}]} \sup_m f_m(x).$$

Suppose $\displaystyle\int_{[0,1]} \sup_n f_n(x) < \infty.$

Then the tail of the above series goes to zero, i.e.

$$\int_{[0,\frac{1}{n}]} \sup_m f_m(x) = \sum_{k=n}^{\infty} \int_{[\frac{1}{k+1}, \frac{1}{k}]} \sup_m f_m(x)$$

$\rightarrow 0$ as $n \rightarrow \infty$. But then there is an n s.t.

$$\frac{1}{2} > \int_{[0,\frac{1}{n}]} \sup_m f_m(x) \geq \int_{[0,\frac{1}{n}]} f_n(x) \geq \frac{1}{2}$$

a contradiction. $\square$