2011, Spring

Incomplete: 2, 3.

Problem 1.

For each $j \in \mathbb{N}$, choose $E_j, F_j \in \mathcal{B}_{\mathbb{R}}$ so $m(A \setminus E_j) \leq m(F_j \setminus E_j) \leq j^{-1}$, and set $E := \bigcup_{j=1}^{\infty} E_j$. Then

$$m(A \setminus E) = m\Big(\bigcup_{j=1}^{\infty}(A \setminus E_j)\Big) = \lim_{j\to\infty} m(A \setminus E_j) \leq \lim_{j\to\infty} \frac{1}{j} = 0.$$

Hence $A = E \sqcup (A \setminus E)$, with $E \in \mathcal{B}_{\mathbb{R}}$ and $A \setminus E$ being m-null. So since m is complete, then $A \in \mathcal{B}_{\mathbb{R}}$ as well. □

Problem 4.

(a) Suppose (w.l.o.g.) that $F_1 \cap F_2 \cap F_3 \cap F_4 = $. Then $\sum_{j=1}^{7} \mathbb{1}_{F_j} \leq 3$ on all of $[0,1]$, whereby

$$3.5 = \sum_{j=1}^{7} \frac{1}{2} \leq \sum_{j=1}^{7} m(F_j) = \int_{[0,1]} \sum_{j=1}^{7} \mathbb{1}_{F_j} \leq 3m([0,1]) = 3,$$

a contradiction. □

(b) Suppose $\int_{[0,1]} \sup_{n\in\mathbb{N}} f_n < \infty$. Since $f_n \geq 0$ for each $n \in \mathbb{N}$, we have

$$\infty > \int_{[0,1]} \sup_{n\in\mathbb{N}} f_n = \sum_{j=1}^{\infty} \int_{[\frac{1}{j+1}, \frac{1}{j}]} \sup_{n\in\mathbb{N}} f_n.$$

Then because the sum on the right-hand side is convergent, we must have

$$0 = \lim_{N\to\infty} \sum_{j=N}^{\infty} \int_{[\frac{1}{j+1}, \frac{1}{j}]} \sup_{n\in\mathbb{N}} f_n = \lim_{N\to\infty} \int_{[0, \frac{1}{N}]} \sup_{n\in\mathbb{N}} f_n \geq \lim_{N\to\infty} \int_{[0, \frac{1}{N}]} f_N \geq \lim_{N\to\infty} \frac{1}{2} = \frac{1}{2},$$

a contradiction. □