105p.1

(1) A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *upper semicontinuous* (or *u.s.c.*) if for all $x \in \mathbb{R}$ and all $\epsilon > 0$ there exists $\delta > 0$ such that $f(y) < f(x) + \epsilon$ whenever $|y - x| < \delta$.

(i) Show that every u.s.c. function is Borel measurable. HINT: Consider $\{x : f(x) < a\}$.

(ii) Suppose μ is a finite measure on $\mathbb{R}$ and A is a closed subset of $\mathbb{R}$. Using (i) or otherwise, show that the function $x \mapsto \mu(x + A)$ is measurable. Here $x + A = \{x + y : y \in A\}$.

(i) Consider $E_a := \{x . f(x) < a\} = f^{-1}((-\infty, a))$

Let $x \in E_a$. Take $\varepsilon = (a - f(x))/2 > 0$.
f u.s.c. $\Rightarrow$ $\exists\ \delta > 0$ s.t. $f(y) < f(x) + \varepsilon < a$ whenever $|y - x| < \delta$.
which is to say $y \in E_a$ as well.
Thus E_a is open and therefore Borel measurable
$\Rightarrow$ f is Borel measurable. $\square$

(ii) Define $f(x) := \mu(x + A) = \mu(\mathbb{R}) - \mu((x+A)^c) = \mu(\mathbb{R}) - \mu(x + A^c)$.
Claim: f is u.s.c., which implies f is Borel measurable by (i).

Let $x \in \mathbb{R}$, $\varepsilon > 0$.

105p.2

(2) Suppose $\{f_n\}$ and f are measurable functions on $(X, \mathcal{M}, \mu)$ and $f_n \to f$ in measure. Is it necessarily true that $f_n^2 \to f^2$ in measure if:

(a) $\mu(X) < \infty$

(b) $\mu(X) = \infty$.

In each case, prove or give a counterexample.

$f_n \to f$ in measure if $\mu(\{x : |f_n(x) - f(x)| > \varepsilon\}) \longrightarrow 0$ as $n \to \infty$, $\forall \varepsilon > 0$.

$$|f_n(x) - f(x)|^2 = f_n(x)^2 - 2f_n(x)f(x) + f(x)^2$$

NTS $\mu(\{x : |f_n(x)^2 - f(x)^2| > \varepsilon\}) \to 0$ as $n \to \infty$, $\forall \varepsilon > 0$.

(a) $\mu(X) < \infty \Rightarrow \mu(\{x : |f_n(x) - f(x)| \leq \varepsilon\}) \longrightarrow \mu(X)$ as $n \to \infty$, $\forall \varepsilon > 0$.

$\Rightarrow \exists N$ s.t. $\forall n \geq N$, $\mu(\{x : |f_n(x) - f(x)| \leq \varepsilon\}) > \mu(X) - \varepsilon'$. $\forall \varepsilon' > 0$.

$100 - 99.5 = .5$

$100^2 - 99.5^2 = 10000 - 9900.25 = 200.$

if $|f_n(x) - f(x)| < 1$ then $0 < f_n(x)^2 - 2f_n(x)f(x) + f(x)^2 < 1$

$f_n(x)^2 - f(x)^2 < 2f_n(x)f(x) - 2f(x)^2 + 1$

$= 2f(x)\underbrace{(f_n(x) - f(x))}_{< \varepsilon} + 1$

10Sp. 3

(3) Suppose $f : [0,1] \to \mathbb{R}$ is a strictly increasing absolutely continuous function. Let m denote Lebesgue measure. If $m(E) = 0$ show that $m(f(E)) = 0$.

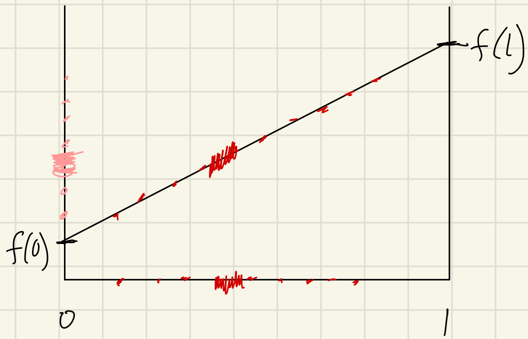
f strictly increasing if $\forall\, x, y \in [0,1]$ with $x < y$, we have $f(x) < f(y)$.

f abs. cont. if $\forall\, \varepsilon > 0$ $\exists\, \delta$ s.t. for any finite collection disjoint intervals $\{(x_i, y_i)\}_1^n$ the following holds;

$$\sum_1^n |y_i - x_i| < \delta \implies \sum_1^n |f(y_i) - f(x_i)| < \varepsilon.$$

$$\Longleftrightarrow \quad f(x) = f(0) + \int_0^x f'(t)\,dt \qquad \text{by Fund. Thm.}$$

$$\begin{aligned}
f(E) &= \{f(x) : x \in E\} \\
&= \{f(0) + \textstyle\int_0^x f'(t)\,dt : x \in E\} \\
&= f(0) + \{\textstyle\int_0^x f'(t)\,dt : x \in E\} \\
m(f(E)) &= m(f(0) + \{\textstyle\int_0^x f'(t)\,dt : x \in E\}) \\
&= m(\{\textstyle\int_0^x f'(t)\,dt : x \in E\}).
\end{aligned}$$



10Sp. 4

(4) For $n \geq 1$ define h_n on $[0,1]$ by

$$h_n = \sum_{j=1}^{n} (-1)^j \chi_{(\frac{j-1}{n}, \frac{j}{n}]}.$$

Here χ_E denotes the characteristic function of E. If f is Lebesgue integrable on $[0,1]$, show that

$$\lim_{n\to\infty} \int_{[0,1]} f h_n \, dm = 0.$$

HINT: First consider f in a suitably smaller function space.

First consider simple functions defined on intervals, i.e. a function $\phi = \sum_1^K \alpha_i \chi_{A_i}$ where $\{A_i\}_1^K$ are disjoint intervals on $[0,1]$.

Then, $\int_0^1 \phi h_n \, dm = \sum_{i=1}^K \int_0^1 \alpha_i \chi_{A_i} h_n \, dm$

$= \sum_{i=1}^K \alpha_i \int_{A_i} h_n \, dm$

Consider $\left| \int_{A_i} h_n \, dm \right| < 2/n$

$\Rightarrow \int_{A_i} h_n \, dm \to 0$ as $n \to \infty$.

$\Rightarrow \int_0^1 \phi h_n \, dm \to 0$ as $n \to \infty$.