Spring '10 :

1. [$f:\mathbb{R}\to\mathbb{R}$ u.s.c if for all $x\in\mathbb{R}$, $\varepsilon>0$, $\exists\delta>0$ s.t $|y-x|<\delta \Rightarrow f(y)<f(x)+\varepsilon$]

(i) Show every u.s.c. function is Borel measurable

(ii) μ finite measure on $\mathbb{R}$, $A\subseteq\mathbb{R}$ closed; show $x\mapsto\mu(x+A)$ is measurable :

(i) Recall that a function $f:\mathbb{R}\to\mathbb{R}$ is Borel measurable if every set $f^{-1}((-\infty,a])=\{f\le a\}$ is a Borel set.

- Consider $x\in f^{-1}((-\infty,a])=\{f\le a\}$ and choose $\varepsilon>0$:

  f is u.s.c., hence $\exists\delta>0$ such that

  $|y-x|<\delta \Rightarrow f(y)<f(x)+\varepsilon$, i.e.:

  $y\in(x-\delta,x+\delta) \Rightarrow f(y)<f(x)+\varepsilon<a+\varepsilon$

- The inequalities are strict, hence letting $\varepsilon\to0$, $\exists\delta'>0$ s.t $y\in(x-\delta',x+\delta')\Rightarrow f(y)\le a$, here $(x+\delta',x+\delta')\subseteq\{f\le a\}$ hence x is contained in an open set contained in $f^{-1}((-\infty,a])$; x was arbitrary, hence $f^{-1}((-\infty,a])$ is open, hence Borel.

(ii) part (i) tells us that u.s.c. $\Rightarrow$ measurable, hence we want to show that $f(x)=\mu(x+A)$ is u.s.c.

- f u.s.c $\Leftrightarrow \limsup_{n\to\infty} f(x_n)\le f(x)$ for $x_n\to x$: Let $x_n\to x$; for $\delta>0$, $\exists M$ s.t. $n\ge M\Rightarrow |x_n-x|<\delta \Rightarrow f(x_n)<f(x)+\varepsilon$
  $\Rightarrow \limsup_{n\to\infty} f(x_n)<f(x)+\varepsilon \Rightarrow \limsup_{n\to\infty} f(x_n)\le f(x)$ as $\varepsilon\to0$ $(\Rightarrow)$

- On the other hand, suppose $\limsup_{n\to\infty} f(x_n)\le f(x)$, then $\lim_{k\to\infty}\left(\sup_{n\ge k} f(x_n)\right)\le f(x)$, hence for k large (say M), we have $\sup_{n\ge M} f(x_n)<f(x)+\varepsilon$, hence $f(x_n)<f(x)+\varepsilon$ for $n\ge M$,
  hence f is u.s.c.

So we want to show $\limsup_{n\to\infty} f(x_n) \le f(x)$, for $x_n \to x$.

Let $f(x) = \mu(x+A)$; now:

$$\limsup_{n\to\infty} f(x_n) = \limsup_{n\to\infty} \mu(x_n+A) \quad (\ast)$$

$\underline{\limsup_{n\to\infty} \mu(A_n) \le \mu(\limsup_{n\to\infty} A_n)}$:

$B_k = \bigcup_{n=k}^{\infty} A_n$, $B_{k+1} \subseteq B_k$

$$\mu(\limsup_{n\to\infty} A_n) = \mu\left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} A_n\right) = \mu\left(\bigcap_{k=1}^{\infty} B_k\right)$$

$\to \mu$ finite so may apply cont from above

$$= \lim_{k\to\infty} \mu(B_k) = \lim_{k\to\infty} \mu\left(\bigcup_{n=k}^{\infty} A_n\right) \ge \lim_{k\to\infty}\left(\sup_{n\ge k} \mu(A_n)\right) = \limsup_{n\to\infty} \mu(A_n).$$

Hence $\limsup_{n\to\infty} \mu(x_n+A) \le \mu(\limsup_{n\to\infty} x_n+A)$;

Suppose $y \in \limsup_{n\to\infty}(x_n+A) = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}(x_k+A)$

$\Rightarrow y \in \bigcup_{k=n}^{\infty}(x_k+A)$ for all n

$\Rightarrow y = x_{k_j} + a_j$ for some $a_j \in A$, for all j

Now, letting $j\to\infty$, we get $y = x + \lim_{j\to\infty} a_j$

Since A is closed, $\lim_{j\to\infty} a_j \in A$, hence $y \in x+A$

so: $\limsup_{n\to\infty}(x_n+A) \subseteq x+A$

$\Rightarrow \mu(\limsup_{n\to\infty}(x_n+A)) \le \mu(x+A)$

$\Rightarrow \limsup_{n\to\infty} \mu(x_n+A) \le \mu(x+A) \Rightarrow \limsup_{n\to\infty} f(x_n) \le f(x)$,

hence u.s.c.,

hence measurable

② $(f_n), f$ measurable on (X,μ), $f_n \to f$ in measure.
Is it true that $f_n^2 \to f^2$ in measure if:

(a) $\mu(X) < \infty$, (b) $\mu(X) = \infty$

(a) f is measurable, hence $A_N = \{|f| > N\}$ is measurable.
see that $A_{N+1} = \{N+1 < |f|\} \subseteq \{N < |f|\} = A_N$, and
$\mu(A_0) = \mu(X) < \infty$, hence we may use continuity from above.
First, let $B_N = \{N < |f| \leq N+1\}$; then $A_N = \bigcup_{i=N}^{\infty} B_i$
and $\sum_{i=0}^{\infty} \mu(B_i) = \mu(\bigcup_{i\geq 0} B_i) = \mu(X) < \infty$. hence

Now: $\lim_{N\to\infty} \mu(A_N) = \mu(\bigcap_{N\geq 0} A_N)$

$\leq \mu(A_k)$ for all k

$= \mu(\bigcup_{i=k}^{\infty} B_i) = \sum_{i=k}^{\infty} \mu(B_i)$ for all k.

Hence let $k \to \infty$ and then $\sum_{i=k}^{\infty} \mu(B_i) \to 0$ since the tail of a convergent series must go to zero.

Hence $\lim_{N\to\infty} \mu(A_N) = 0$. hence we have that, for $\varepsilon > 0$,
$\exists M \geq 0$ such that $\mu\{|f_n| \geq M\} < \varepsilon$, $\mu\{|f| \geq M\} < \varepsilon$.

- Now see the following: $|f_n^2 - f^2| = |f_n^2 - ff_n + ff_n - f^2|$

$\leq |f_n^2 - ff_n| + |ff_n - f^2|$

$= |f_n| \cdot |f_n - f| + |f| |f_n - f|$

$\leq + f_n$

Now:

$$\{|f_n^2-f^2|\geq\varepsilon\}\subseteq\{|f_n||f_n-f|+|f||f_n-f|\geq\varepsilon\}$$
$$\subseteq\{|f_n||f_n-f|\geq\tfrac{\varepsilon}{2}\}\cup\{|f||f_n-f|\geq\tfrac{\varepsilon}{2}\}$$
$$\subseteq\{|f_n|\geq M\}\cup\{|f_n-f|\geq\tfrac{\varepsilon}{2M}\}\cup\{|f|\geq M\}\cup\{|f_n$$
$$\cup\{|f_n-f|\geq\tfrac{\varepsilon}{2M}\}$$

Hence:

$$\mu\{|f_n^2-f^2|\geq\varepsilon\}\leq\mu\{|f_n|\geq M\}+\mu\{|f_n-f|\geq\tfrac{\varepsilon}{2M}\}+\mu\{|f|\geq M\}+\mu\{|f_n-f|\geq\tfrac{\varepsilon}{2M}$$
$$<\varepsilon+\varepsilon+\mu\{|f_n-f|\geq\tfrac{\varepsilon}{2M}\}+\mu\{|f_n-f|\geq\tfrac{\varepsilon}{2M}\}$$

$\longrightarrow$ by part 1

$$<\varepsilon+\varepsilon+\varepsilon+\varepsilon$$

for n large enough $\longrightarrow$ since $f_n\to f$ converges in measure

Therefore, $f_n^2\to f^2$ in measure.

(b) $\mu(X)=\infty$; let $X=\mathbb{R}$, $\mu=m$ Lebesgue.

Let $f(x)=x^2$, $f_n(x)=x^2+\frac{1}{n}$; then, for $x\in[n,\infty)$,

we have: $|f_n(x)^2-f(x)^2|=|(x^2+\frac{1}{n})^2-x^4|$
$$=|x^4+\frac{2x^2}{n}+\frac{1}{n^2}-x^4|$$
$$=|\frac{2x^2}{n}+\frac{1}{n^2}|=\frac{2x^2}{n}+\frac{1}{n^2}$$ since all positive

So for $x\in[n,\infty)$,

$|f_n^2(x)-f^2(x)|=\frac{2x^2}{n}+\frac{1}{n^2}\geq\frac{2x^2}{n}\geq 2n\geq 2$,

hence $[n,\infty)\subseteq\{|f_n^2-f^2|\geq 2\}$, hence

$m([n,\infty))\leq m\{|f_n^2-f^2|\geq 2\}$, but $\lim\limits_{n\to\infty}m([n,\infty))=\infty$

since m is an infinite measure and $m([n,\infty))=\infty$.

Therefore $f_n^2\not\to f^2$ in measure.

③ $f:[0,1]\to\mathbb{R}$ strictly increasing, absolutely continuous
~~m~~ Lebesgue measure; Show $m(E)=0 \Rightarrow m(f(E))=0$

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f strictly increasing, hence 1-1, hence f^{-1} is well-defined. Now:

$$m(f(E)) = \int_{\mathbb{R}} \chi_{f(E)}\,dx = \int \chi_E(f^{-1}(x))\,dx$$

Now let $u=f^{-1}(x) \Rightarrow du=(f^{-1})'(x)\,dx$

But recall $(f\circ f^{-1})(x)=x$, hence $(f\circ f^{-1})'(x)=1$

$$\Rightarrow f'(f^{-1}(x))(f^{-1})'(x)=1 \Rightarrow (f^{-1})'(x)=\frac{1}{f'(f^{-1}(x))}=\frac{1}{f'(u)}$$

Hence $du=\dfrac{dx}{f'(f^{-1}(x))} \Rightarrow f'(u)\,du=dx$

So $\int \chi_E(f^{-1}(x))\,dx = \int \chi_E(u)\,f'(u)\,du = \int_E f'(u)\,du = 0$

since $m(E)=0$, hence $\underline{m(f(E))=0}$.

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④ $n\geq 1$; define h_n on $[0,1]$ by $h_n=\sum_{j=1}^{n}(-1)^j\chi_{[\frac{j-1}{n},\frac{j}{n}]}$
Show $\lim_{n\to\infty}\int_{[0,1]} fh_n\,dm=0$ for $f\in L^1(\mathbb{R})$

- $\underline{\text{case } f=\chi_{(a,b)}}$: $\int_{[0,1]} fh_n\,dm=\int_{(a,b)} h_n\,dm$

Now let $I_j=[\frac{j-1}{n},\frac{j}{n}]$; clearly $[0,1]=\bigcup_{j=1}^{n} I_j$, hence let $I_j, I_{j+1},\dots, I_{j+(k-1)}$ be the k intervals $\underline{\text{completely}}$ contained in (a,b). (clearly, k may be 0).

Finally, let J_1, J_2 be the I_j s.t. $J_1\cap(a,b)=(a,a+\delta_1)$ and $J_2\cap(a,b)=(b-\delta_2,b)$ for some $\delta_1,\delta_2\geq 0$.

Then: $(a,b)=J_1\cup J_2\cup\left(\bigcup_{i=0}^{k-1} I_{j+i}\right)$ since the I_j are disjoint.

So now:

$$\int_{(a,b)} h_n\,dm = \int_{J_1} h_n\,dm + \int_{J_2} h_n\,dm + \sum_{i=0}^{k-1} \int_{I_{j+i}} h_n\,dm$$

Since for all $x \in I_j$, $h_n(x) = (-1)^j$, we have that

$$\int_{I_j} h_n\,dm = \frac{(-1)^j}{n}, \text{ and}$$

hence: $\underline{-\frac{1}{n} < \int_{J_i} h_n < \frac{1}{n}}$ for J_1, J_2

and $\displaystyle\sum_{i=0}^{k-1} \int_{I_{j+i}} h_n\,dm = \sum_{i=0}^{k-1} \frac{(-1)^{j+i}}{n} = \begin{cases} 0 & \text{if } k \text{ even} \\ \pm\frac{1}{n} & \text{if } k \text{ odd.} \end{cases}$

hence $\underline{-\frac{1}{n} < \sum_{i=0}^{k-1} \int_{I_{j+i}} h_n\,dm < \frac{1}{n}}$ also

<u>Therefore</u>: $-\frac{3}{n} < \int_{(a,b)} h_n\,dm < \frac{3}{n}$, hence

$\lim_{n\to\infty} \int_{(a,b)} h_n\,dm = 0$ by the squeeze theorem, hence

$\lim_{n\to\infty} \int f h_n\,dm = 0$ for $f = \chi_{(a,b)}$.

- <u>Case $f = \sum c_i \chi_{(a_i,b_i)}$</u>, i.e. an int. simple fn. with E_i = interval

Then: $\lim_{n\to\infty} \int f h_n\,dm = \lim_{n\to\infty} \int \left(\sum_{i=1}^{n} c_i \chi_{(a_i,b_i)}\right) h_n\,dm$

$= \lim_{n\to\infty} \sum_{i=1}^{n} c_i \int \chi_i h_n\,dm = \sum_{i=1}^{n} c_i \left(\lim_{n\to\infty} \int \chi_i h_n\,dm\right) = \sum_{i=1}^{n} c_i(0) = 0$

- <u>general case</u>: for $\varepsilon > 0$, $\exists$ integrable simple fn. with E_i = finite unions of intervals ϕ s.t. $\int |f-\phi| < \varepsilon$ (since Lebesgue

Then: $\varepsilon > \int |f-\phi| = \int |h_n||f-\phi| \geq \left|\int h_n f - h_n\phi\right| \geq \left|\int h_n f\right| - \left|\int h_n \phi\right|$

hence as $n \to \infty$, $\left|\int h_n f\right| < \varepsilon$, but since ε was arbitrary,

$\underline{\left|\int h_n f\right| = 0 \text{ as } n \to \infty.}$

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$\sum (f(x_i)-f(y_i)) < \frac{\epsilon}{10}$

(3.) $f:[0,1]\to\mathbb{R}$ strictly increasing, abs. cont.
Show $m(E)=0 \Rightarrow m(f(E))=0$

Since $E\subseteq[0,1]$ is measure 0, $\exists$ countable sequence of intervals (x_i,y_i) s.t. $E\subseteq\bigcup_{i=1}^{\infty}(x_i,y_i)$
and $m(\bigcup_{i=1}^{\infty}(x_i,y_i)) \leq \sum_{i=1}^{\infty}|y_i-x_i| < \delta$, for any given $\delta>0$.

Now, choose $\varepsilon>0$, since f is abs. cont., $\exists\delta>0$
s.t. $\sum_{i=1}^{n}|y_i-x_i|<\delta \Rightarrow \sum_{i=1}^{n}|f(y_i)-f(x_i)|<\varepsilon$.

See that: $f(E)\subseteq f(\bigcup_{i=1}^{\infty}(x_i,y_i))\subseteq\bigcup_{i=1}^{\infty}(f(x_k),f(y_k))$
since f is continuous and strictly increasing.
Then we have:
$$m(f(E))\leq m\left(\bigcup_{i=1}^{\infty}(f(x_k),f(y_k))\right) \leq \sum_{i=1}^{\infty}|f(y_k)-f(x_k)| < \varepsilon$$
since we chose intervals s.t. $\sum|y_k-x_k|<\delta$

$m(E)=0$, $\exists$ countable seq $\bigcup_{i=1}^{\infty}(x_i,y_i)\supseteq E$
s.t.
$m(\bigcup(x_i,y_i)\setminus E)<\varepsilon$
$\parallel$
$m(\bigcup(x_i,y_i))-m(E)$ with $m(E)=0$

$(x_i,y_i)\subseteq\bigcup(x_i,y_i)$
$\to m((x_i,y_i))\leq m(\bigcup(x_i,y_i))<\varepsilon$
$m((x_i,y_i))<\varepsilon$