Real, Spring 2010

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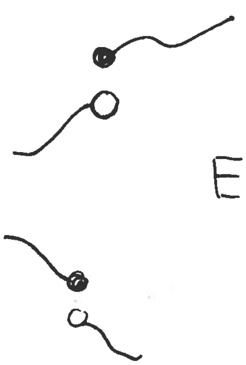
(1) $f : \mathbb{R} \to \mathbb{R}$ upper semicontinuous



if $x_n \to x \Rightarrow \limsup_{x_n \to x} f(x_n) \leq f(x)$.

Equivalently, for each x and $\varepsilon > 0$ there exists $\delta > 0$ s.t. $f(y) - f(x) < \varepsilon$ for all $|x - y| < \delta$.

(i) show f u.s.c. $\Rightarrow$ f Borel.

It suffices to show $f^{-1}(\{x : x < a\})$ is open for all $a \in \mathbb{R}$. Let $f(x) < a$ and set $\varepsilon = a - f(x)$. Choose $\delta > 0$ by the def of u.s.c. so that $|x - y| < \delta \Rightarrow$ $f(y) - f(x) < \varepsilon = a - f(x)$ i.e. $f(y) < a$ i.e. $y \in f^{-1}(\{x : x < a\})$ i.e. this set is open. $\square$

Suppose μ is finite Borel on $\mathbb{R}$

and A is closed subset of $\mathbb{R}$.

So $x \mapsto \mu(x+A)$ defines a function

$f: \mathbb{R} \to \mathbb{R}$. Show f is Borel.

We show f is u.s.c. by showing

$\limsup_{x_n \to x} f(x_n) \leq f(x)$. We use Fatou.

Observe $-\limsup_n \mu(x_n+A) = -\limsup_n \int \chi_{x_n+A}$

$$= \liminf_n \int -\chi_{x_n+A} \overset{\text{FATOU}}{\geq} \int \liminf_n -\chi_{x_n+A} =$$

$$- \int \limsup_n \chi_{x_n+A} \overset{?}{\geq} - \int \chi_{x+A} =$$

$$-\mu(x+A) \implies \mu(x+A) \geq \limsup_n \mu(x_n+A).$$

It remains to answer "?" for which it suffices to show $\limsup_n \chi_{x_n+A} \leq \chi_{x+A}$.

This is equivalent to saying for all $y \notin x+A$ that $y \in x_n + A$ for only finitely many n. Assume otherwise. Then let $\varepsilon > 0$ and let N be s.t. $|x_n - x| < \varepsilon$ for all $n \geq N$ and take $n \geq N$ s.t. $y \in x_n + A$, say $y = x_n + a$. Then $|y - (x+a)| = |x - x_n| < \varepsilon$. So y is a limit point of $x+A$ which is closed, so $y \in x+A$, violating our assumption. Hence, $\limsup_n \chi_{x_n+A} \leq \chi_{x+A}$ and f is u.s.c. hence Borel by (i). $\square$

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(2) If $f_n \to f$ in measure, show

(i) $m(X) < \infty \implies f_n^2 \to f^2$ in measure,

(ii) $m(X) = \infty \implies f_n^2 \not\to f^2$ in measure, necessarily.

(i) $\lim_{n \to \infty} m(\{x : |f(x) - f_n(x)| \geq \varepsilon\}) = 0 \quad \forall \varepsilon > 0.$

Since $m(X) < \infty$, $\lim_{M \to \infty} m(\{x : |f(x)| > M\}) = 0$.

Fix $\delta, \varepsilon > 0$. NTS $\exists N$ s.t. $n \geq N$

$\Rightarrow m(\{x : |f^2(x) - f_n^2(x)| > \delta\}) < \varepsilon$.

$$|f^2(x) - f_n^2(x)| = |ff - ff_n + ff_n - f_n f_n|$$

$$\leq |f||f - f_n| + |f_n||f - f_n|$$

so $\{|f^2 - f_n^2| > \delta\} \subset \{|f||f - f_n| > \frac{\delta}{2}\} \cup$

$\{|f_n||f - f_n| > \frac{\delta}{2}\}$.

Since $\lim_{M \to \infty} m(\{|f| > M\}) = 0$ choose $M > 0$ s.t.

$m(\{|f| > M\}) < \varepsilon$.

So $\{|f||f-f_n| > \frac{\delta}{2}\} = \{|f-f_n| > \frac{\delta}{2M}\} \cup$

$\{|f||f-f_n| > \frac{\delta}{2} \ \& \ |f| > M\}$. Taking

measures, the measure of $\{|f||f-f_n| > \frac{\delta}{2}\}$

is $\leq 2\varepsilon$ taking n sufficiently large.

Note also $\{|f_n| > M\} \subset \{|f-f_n| > \frac{M}{2}\} \cup \{|f| > \frac{M}{2}\}$

so for sufficiently large M & n

$\{|f_n| > M\} < \varepsilon$ and by the same argument

as above $\mu\{|f_n||f-f_n| > \frac{\delta}{2}\} \leq 2\varepsilon$, possibly increasing n once again. And we are done. $\square$

(ii) Take $f_n(X) = X + \frac{1}{n}$ and $f(X) = X$ on $\mathbb{R}^{\geq 0}$. Then $|f_n(X) - f(X)| = \frac{1}{n} < \varepsilon$ for $n \gg 1$

hence $f_n \to f$ in measure.

Now, $\left|f^2(x) - f_n^2(x)\right| = \left|x^2 - \left(x^2 + \frac{2x}{n} + \frac{1}{n^2}\right)\right|$

recall $x \geq 0$

$$= \frac{2x}{n} + \frac{1}{n^2} \geq \frac{2x}{n}.$$

But for $x \geq n^2$ then,

$$\left|f^2(x) - f_n^2(x)\right| \geq 2n \geq 2.$$

So for $n \geq 1$, $m\left(\{x : \left|f^2(x) - f_n^2(x)\right| \geq 2\}\right)$

$$\geq m\left(\{x : x \geq n^2\}\right) = \infty.$$

So $f_n^2 \not\to f^2$ in measure. $\square$

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(3). $f : [0,1] \to \mathbb{R}$ strictly increasing absolutely continuous. Show if $m(E) = 0$, then $m(f(E)) = 0$.

Since f is strictly increasing and $[0,1]$ is compact and f is continuous, f is a homeomorphism from $[0,1]$ onto its image. So $f(E)$ is Borel.

~~Since f is continuous it is measurable.~~

~~Since f is strictly increasing,~~

~~$f([0,1]) \subset [f(0), f(1)]$ (in fact, an equality).~~

~~So f is bounded, hence is integrable on $[0,1]$.~~

We use the COV theorem for absolutely continuous transformations.

Thm Let $f : [a,b] \longrightarrow \mathbb{R}$ be increasing and absolutely continuous, and let $g : [f(a), f(b)] \to \mathbb{R}$ be integrable. Then

$$\int_{f(a)}^{f(b)} g(y)\,dy = \int_a^b g(f(x))\, f'(x)\,dx$$

In our situation, let $g(y) = \chi_{f(E)}(y)$ on $[f(0), f(1)]$. Since $f(E)$ is Borel, $g(y)$ is measurable, hence is integrable on $[f(0), f(1)]$. Observe $g(f(x))$

$$= \chi_{f(E)}(f(x)) = \begin{cases} 1 & \text{if } f(x) \in f(E) \\ 0 & \text{if } f(x) \notin f(E) \end{cases}$$

$= \chi_E(x)$. By the theorem, we conclude

$$m(f(E)) = \int_{f(0)}^{f(1)} g(y)\,dy = \int_0^1 g(f(x))\, f'(x)\,dx$$

$$= \int_0^1 \chi_E(x) f'(x)\,dx = \int_E f'(x)\,dx = 0$$

since $m(E) = 0$. $\square$

(4) For $n \geq 1$ define h_n on $[0,1]$ by $h_n = \sum_{j=1}^{n} (-1)^j \chi_{\left(\frac{j-1}{n}, \frac{j}{n}\right]}$ and $h_n(0) = 0$ (or anything). If f is integrable on $[0,1]$ show $\lim_{n \to \infty} \int f h_n = 0$ (*).

It suffices to prove the result for when f is χ_E. Indeed, By linearity, (*) is true for simple functions. ~~If $f \geq 0$ and $\phi_n \leq f$ are simple functions increasing to~~

Then, Fix $\varepsilon > 0$. we show $\left| \int f h_n \right| \to \leq \varepsilon$ as $n \to \infty$. Since f is integrable, choose a simple function ϕ s.t. $\int |f - \phi| < \varepsilon$. Then $\left| \int f h_n \right| =$

$$\left| \int (f - \phi) h_n + \int \phi h_n \right| \leq \int |f - \phi| + \left| \int \phi h_n \right| \leq$$

$$\varepsilon + \left| \int \phi h_n \right| \longrightarrow \varepsilon + 0 \text{ as } n \to \infty$$

since ϕ is simple. We conclude $\lim_{n \to \infty} \int f h_n = 0$.

It remains to prove $\lim_{n\to\infty} \int \chi_E h_n = 0$.

First, observe this is immediate if $E = (a,b)$; for in the integral $\int \chi_{(a,b)} h_n$, the most area that doesn't cancel out is $\frac{1}{n}$, hence

$$\left| \int \chi_{(a,b)} h_n \right| \le \frac{1}{n} \to 0 \text{ as } n \to \infty.$$

It follows that if $U = \bigcup_j (a_j, b_j)$ disjoint, then $\lim_{n\to\infty} \int \chi_U h_n = 0$. Indeed, this is

$$\lim_{n\to\infty} \int \sum_j \chi_{(a_j,b_j)} h_n = \lim_{n\to\infty} \sum_j \int \chi_{(a_j,b_j)} h_n$$

$$\left(\text{because } \sum_j \int \left| \chi_{(a_j,b_j)} h_n \right| = \sum_j |b_j - a_j| < \infty\right)$$

$$= \sum_j \lim_{n\to\infty} \int \chi_{(a_j,b_j)} h_n = 0$$

(because if $F_n(j) = \int \chi_{(a_j,b_j)} h_n$, then we have shown $F_n(j) \to 0$ as $n \to \infty$, and also $\left| \int \chi_{(a_j,b_j)} h_n \right| \le \int \chi_{(a_j,b_j)} = |b_j - a_j|$ and $\sum_j |a_j - b_j| < \infty$, hence

$F(j) = |b_j - a_j|$ is a dominating function for the $F_n(j)$; hence, by integrating in counting measure over the j we have

$$\lim_{n\to\infty} \sum_j \int \chi_{(a_j,b_j)} h_n = \sum_j \lim_{n\to\infty} \int \chi_{(a_j,b_j)} h_n$$
$$= \sum_j 0 = 0. \Big)$$

Finally, we use upper regularity to show $\lim_{n\to\infty} \int \chi_E h_n = 0$ for any E.

<u>Lemma</u> If f_n is uniformly bounded and $\lim_{n\to\infty} \int_a^b \chi_u f_n = 0$ for all $U \supset E$, then $\lim_{n\to\infty} \int_a^b \chi_E f_n = 0$.

pf/ Fix $\varepsilon > 0$. By upper regularity, there exists $U = \bigcup_j (a_j, b_j)$ disjoint s.t. $m(U) - m(E) < \varepsilon$.

Since $U, E \subset [a,b]$ is finite measure, $m(U - E) < \varepsilon$.

So $\left|\int \chi_E f_n\right| = \left|\int \chi_U f_n - \int \chi_{U-E} f_n\right|$

$\leq \left|\int \chi_U f_n\right| + M\varepsilon \quad (|f_n| \leq M)$

$\longrightarrow \leq M\varepsilon$ as $n \to \infty$.

Therefore, $\lim_{n\to\infty} \int \chi_E f_n = 0$. ✓

Applying the lemma to $f_n = h_n$, we are done.

$\square$