## 2010, Spring

**Problem 1.**

(i) Let $f$ be u.s.c. and $a \in \mathbb{R}$. If $x_0 \in f^{-1}((-\infty, a)) = \{x \in \mathbb{R} \mid f(x) < a\}$, then $f(x_0) + \epsilon < a$ for some $\epsilon > 0$. Then there's some $\delta > 0$ so that $f(x) < f(x_0) + \epsilon < a$ whenever $|x - x_0| < \delta$. Thus $f^{-1}((-\infty, a))$ is open, and in particular Borel. Since sets of the form $(-\infty, a)$ for $a \in \mathbb{R}$ generate $\mathcal{B}_{\mathbb{R}}$, this shows that $f$ is measurable. □

(ii) We first claim that a map $f : \mathbb{R} \to \mathbb{R}$ is u.s.c. if for each $x \in \mathbb{R}$ we have $\limsup_{j\to\infty} f(x_j) \leq f(x)$ whenever $\{x_j\}_{j=1}^{\infty} \subset \mathbb{R}$ satisfies $\lim_{j\to\infty} x_j = x$. (In fact, this is an equivalent definition of upper semicontinuity.)

To establish this, suppose $f$ is u.s.c., but there's some $x \in \mathbb{R}$ and a sequence $\{x_j\}_{j=1}^{\infty} \subset \mathbb{R}$ converging to $x$, with $f(x) < a := \limsup_{j\to\infty} f(x_j)$. Let $\epsilon > 0$ be such that $f(x) < a - \epsilon$. By definition of $a$, there's a subsequence $\{x_{j_k}\}_{k=1}^{\infty}$ of $\{x_j\}_{j=1}^{\infty}$ converging to $a$, so all but finitely many of the $x_{j_k}$'s belong to $E := \{y \in \mathbb{R} \mid f(y) \geq a - (\epsilon/2)\}$. By inspection, $E$ is closed, so $x = \lim_{k\to\infty} x_{j_k} \in E$, and hence $a - (\epsilon/2) \leq f(x) < a - \epsilon$, which is impossible.

Now, define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) := \mu(x + A)$. It's enough to show that $f$ satisfies the above condition. Let $\{x_j\}_{j=1}^{\infty} \subset \mathbb{R}$ converge to some $x \in \mathbb{R}$. Since $|f| \leq \mu(\mathbb{R}) < \infty$ on all of $\mathbb{R}$, then

$$\limsup_{j\to\infty} f(x_j) = \limsup_{j\to\infty} \mu(x_j + A) \leq \mu\Big(\limsup_{j\to\infty}(x_j + A)\Big)$$

by reverse Fatou's lemma. By definition of $\limsup$, if $y \in \limsup_{j\to\infty}(x_j + A)$, then $y \in x_j + A$ for infinitely many $j \in \mathbb{N}$. Passing to a subsequence of $\{x_j\}_{j=1}^{\infty}$ if necessary, w.l.o.g. $y = x_j + a_j$, for some $a_j \in A$, for all $j \in \mathbb{N}$, and passing to another subsequence if necessary, w.l.o.g. $\lim_{j\to\infty} a_j$ exists and belongs to $A$ since $A$ is closed. Then $y = x + \lim_{j\to\infty} a_j \in x + A$, whereby we've shown that $\limsup_{j\to\infty}(x_j + A) \subset x + A$. So

$$\limsup_{j\to\infty} f(x_j) \leq \mu\Big(\limsup_{j\to\infty}(x_j + A)\Big) \leq \mu(x + A) = f(x),$$

and this completes the proof. □

**Problem 2.**

(a) **True.** Let $\delta, \epsilon > 0$. Since $\mu(X) < \infty$, there's $M > 0$ large enough so that if $E := \{|f| < M\}$, then $\mu(E^{\mathsf{c}}) < \epsilon/3$. Now $\left|f_n^2 - f^2\right| \leq \left|f_n^2 - f_n f\right| + \left|f_n f - f^2\right| = |f_n| \cdot |f_n - f_n| + |f| \cdot |f_n - f|$, so

$$\left\{\left|f_n^2 - f^2\right| > \delta\right\} \subset \left\{|f_n| \cdot |f_n - f| > \frac{\delta}{2}\right\} \cup \left\{|f| \cdot |f_n - f| > \frac{\delta}{2}\right\}.$$

Thus $\mu\left(E \cap \left\{\left|f_n^2 - f^2\right| > \delta\right\}\right)$ is bounded above by

$$\mu\left(E \cap \left\{|f_n| \cdot |f_n - f| > \frac{\delta}{2}\right\}\right) + \mu\left(E \cap \left\{|f| \cdot |f_n - f| > \frac{\delta}{2}\right\}\right) + \underbrace{\mu(E^{\mathsf{c}})}_{<\epsilon/3}.$$

For large enough $n$ the second term gives

$$\mu\left(E \cap \left\{|f| \cdot |f_n - f| > \frac{\delta}{2}\right\}\right) < \mu\left(\left\{M|f_n - f| > \frac{\delta}{2}\right\}\right) < \frac{\epsilon}{3}.$$

Moreover $|f_n| \cdot |f_n - f| \leq (|f| + |f - f_n|)\,|f - f_n| = |f| \cdot |f_n - f| + |f_n - f|^2$ and so for large enough $n$ the first term gives

$$\mu\left(E \cap \left\{|f_n| \cdot |f_n - f| > \frac{\delta}{2}\right\}\right) \leq \mu\left(E \cap \left\{|f| \cdot |f \cdot f_n| > \frac{\delta}{4}\right\}\right) + \mu\left(\left\{|f_n - f|^2 > \frac{\delta}{4}\right\}\right)$$
$$\leq \mu\left(\left\{M|f - f_n| > \frac{\delta}{4}\right\}\right) + \mu\left(\left\{|f_n - f| > \frac{\delta^{1/2}}{2}\right\}\right) < \frac{\epsilon}{6} + \frac{\epsilon}{6} = \frac{\epsilon}{3}.$$

Hence $\mu\left(E \cap \left\{\left|f_n^2 - f^2\right| > \delta\right\}\right) < \epsilon$. □

(b) **False**. Set $X := (0, \infty)$ with Lebesgue measure $\mu$. If $f_n(x) := x - n^{-1}$ and $f(x) := x$, then for any $\delta > 0$, we have $\mu(\{|f_n(x) - f(x)| > \delta\}) = \mu(\{n^{-1} > \delta\}) \to 0$ and hence $f_n \to f$ in measure. However for any $n \in \mathbb{N}$ and any $x$ in the measure-$\infty$ set $[n, \infty)$,

$$\left|f_n^2(x) - f^2(x)\right| = \left|\left(x^2 - \frac{2x}{n} + \frac{1}{n^2}\right) - x^2\right| = \frac{2x}{n} - \frac{1}{n}^2 \geq 2,$$

whereby $f_n^2 \not\to f^2$ in measure. □

**Problem 3.**

Let $E \subset [0,1]$ have $m(E) = 0$, and let $\epsilon > 0$. Since $f$ is absolutely continuous, there's some $\delta > 0$ such that for any disjoint collection $\{(a_j, b_j)\}_{j=1}^N$, we have

$$\sum_{j=1}^N (b_j - a_j) < \delta \implies \sum_{j=1}^N [f(b_j) - f(a_j)] < \epsilon.$$

By outer regularity of $m$, there's an open set $U \subset [0,1]$ with $E \subset U$ and $m(U) < \delta$. We may write $U$ as a disjoint union $U = \bigsqcup_{j \in J} (a_j, b_j)$ for some countable set $J$. Then for any $N \leq |J|$,

$$\sum_{j=1}^N (b_j - a_j) \leq \sum_{j \in J} (b_j - a_j) = m(U) < \delta \implies \sum_{j=1}^N [f(b_j) - f(a_j)] < \epsilon,$$

and hence it follows that

$$m(f(E)) = m\Big(\bigcup_{j \in J} (f(a_j), f(b_j))\Big) = \sum_{j \in J} [f(b_j) - f(a_j)] \leq \epsilon,$$

where the first inequality used that $f$ was strictly increasing. Hence $m(f(E)) = 0$. □

**Problem 4.**

- Let $f \in \mathsf{L}^1([0,1])$ and choose any $\epsilon > 0$. We may find a simple function $\varphi = \sum_{k=1}^m a_k \mathbb{1}_{E_k}$ with $\|f - \varphi\|_{\mathsf{L}^1([0,1])} < \epsilon$, where $\{a_k\}_{k=1}^m \subset \mathbb{R}$ and $\{E_k\}_{k=1}^m \subset \mathcal{B}_{[0,1]}$ is a disjoint collection of sets. By discarding countably many singletons if necessary, w.l.o.g. $E_k$ is a disjoint union of intervals for each $1 \leq k \leq m$. We further assume w.l.o.g. that $E_k$ is a single interval for each $1 \leq k \leq m$. For each $n \in \mathbb{N}$,

$$\left|\left|\int h_n f\right| - \left|\int h_n \varphi\right|\right| \leq \left|\int h_n (f - \varphi)\right| \leq \int \underbrace{|h_n|}_{=1} |f - \varphi| < \epsilon,$$

  so if the result holds for simple functions which are linear combinations of indicators of intervals, then taking the limit as $n \to \infty$ on each side gives $\mathsf{lim}_{n \to \infty} \left|\int h_n f\right| < \epsilon$. Thus we've reduced to the case of simple functions of this form.

- Now suppose $\varphi = \sum_{k=1}^{m} a_k \mathbb{1}_{E_k}$ is a linear combination of indicators of intervals $E_k \in \mathcal{B}_{[0,1]}$, $1 \leq k \leq m$. If the result holds for indicators of intervals, then

$$\lim_{n\to\infty} \int h_n \varphi = \sum_{k=1}^{m} a_k \underbrace{\lim_{n\to\infty} \int h_n \mathbb{1}_{E_k}}_{=0} = 0,$$

  so we've further reduced to the case of indicators of intervals.

- Finally, let $E \in \mathcal{B}_{[0,1]}$ be an arbitrary interval, fix $n \in \mathbb{N}$, and let $F_{j_1}, \ldots, F_{j_\ell}$ be those intervals $F_j := (\frac{j-1}{n}, \frac{j}{n}]$ with $F_j \subset E$ (w.l.o.g. $j_1 < \cdots < j_\ell$). Setting $G_0 := F_{j_1 - 1}$ and $G_1 := F_{j_\ell + 1}$, then $E \subset G_0 \cup F_{j_1} \cup \cdots \cup F_{j_\ell} \cup G_1$, so

$$\left| \int_{[0,1]} h_n \mathbb{1}_E \right| = \left| \int_E h_n \right| \leq \underbrace{\int_{G_0} 1}_{=1/n} + \left| \sum_{r=1}^{\ell} \int_{F_{j_r}} h_n \right| + \underbrace{\int_{G_1} 1}_{=1/n} = \frac{2}{n} + \left| \sum_{r=1}^{\ell} \frac{(-1)^{j_r}}{n} \right|.$$

  The summands on the right alternate signs as $r$ increases, so the entire sum is either 0 or $\pm 1/n$ depending on the parity of $\ell$. Whichever is the case,

$$\lim_{n\to\infty} \left| \int_{[0,1]} h_n \mathbb{1}_E \right| \leq \lim_{n\to\infty} \left( \frac{2}{n} + \frac{1}{n} \right) = 0.$$

This completes the proof. $\square$