09Sp. 1

a) $(X, \mathcal{B}, \mu)$ μ finite. $\mathcal{A} \subset \mathcal{B}$ an algebra.

$E \in \mathcal{B}$ approx. from inside by $\mathcal{A}$ if $\forall \varepsilon > 0$, $\exists A \in \mathcal{A}$ with $A \subset E$, $\mu(E \setminus A) < \varepsilon$.

Claim: $C = \{E \in \mathcal{B} ; E \text{ approx. from inside by } \mathcal{A}\}$ closed under countable unions.

Let $\{E_i\}_1^\infty$ be approx. from inside by $\mathcal{A}$.

$\Rightarrow$ $\forall \varepsilon > 0$, $\exists A_i \in \mathcal{A}$ with $A_i \subset E_i$, $\mu(E_i \setminus A_i) < \varepsilon/2^i$ $\forall i$.

Is $\bigcup_1^\infty E_i$ approx. from inside by $\mathcal{A}$? Fix $\varepsilon > 0$.

Def $F_i = E_i \setminus \bigcup_{j=1}^{i-1} E_j$ so that $\bigcup_1^\infty F_i = \bigcup_1^\infty E_i$ and F_i disjoint.

$$\mu\left(\bigcup_1^\infty F_i\right) = \underbrace{\sum_1^\infty \mu(F_i)}_{\text{call this } M} \leq \mu(X) < \infty.$$

$$\sum_1^\infty \mu(F_i) = \lim_{K \to \infty} \sum_{i=1}^K \mu(F_i) = M$$

$\Rightarrow$ $\exists K \in \mathbb{N}$ s.t. $\sum_{i=1}^K \mu(F_i) > M - \varepsilon$.

Define $A := \bigcup_1^K A_i \in \mathcal{A}$ since $\mathcal{A}$ is an Algebra.

$$\begin{aligned} \mu\left(\bigcup_1^\infty E_i \setminus A\right) &= \sum_1^K \mu(F_i \setminus A_i) + \mu\left(\bigcup_{K+1}^\infty F_i \setminus A\right) \\ &\leq \varepsilon + \varepsilon \\ &= 2\varepsilon \end{aligned}$$

And $A \subset \bigcup_1^\infty E_i$ $\Rightarrow$ $\bigcup_1^\infty E_i$ approx. from inside by $\mathcal{A}$. $\square$

09Sp. 1

b) Find an example which shows C need not be closed under complements.

$C = \{E \in \mathcal{B} \,;\, E \text{ approx from inside by } A\}$

Let $X = [0,1]$, $E = [0,1] \cap \mathbb{Q}$.
With Lebesgue measure, we have $m(E) = 0$.
Let $A := \{\emptyset, X\}$, which is an algebra.
$\forall \varepsilon > 0$, $\emptyset \subset E$ and $m(E \setminus \emptyset) = 0 < \varepsilon$, so
E is approx. from the inside by A.
But $E^c = [0,1] \setminus \mathbb{Q}$ has measure $m(E^c) = 1$.
And $X \not\subset E^c$ and $m(E^c \setminus \emptyset) = 1 \not< \varepsilon$ for some $\varepsilon > 0$.
$\Rightarrow$ E^c not approx. from inside by A.

$\square$

09Sp. 2

f, g abs. cont. on $[a,b]$
$\Rightarrow$ $\forall \varepsilon > 0$, $\exists\, \delta$ s.t. $\forall$ finite unions of disjoint intervals
$(a_1, b_1), \ldots, (a_N, b_N)$ on $[a,b]$, we have
$$\Sigma_1^N\, b_i - a_i < \delta \Rightarrow \Sigma_1^N |f(b_i) - f(a_i)| < \varepsilon.$$
This holds for g too. Since $[a,b]$ compact, f, g bounded, say by M.

a) Claim: (fg) abs. cont. Induct on N.
First, for $N=1$, $\varepsilon > 0$, we want to show $\exists\, \delta$ s.t.
for any interval $(a_0, b_0) \subset [a,b]$, we have
$$b_0 - a_0 < \delta \Rightarrow |(fg)(b_0) - (fg)(a_0)| < \varepsilon.$$
f, g abs. cont. $\Rightarrow$ $\exists\, \delta_f, \delta_g$ s.t. for any interval $(a_0, b_0) \subset [a,b]$
$$b_0 - a_0 < \delta_f \Rightarrow |f(b_0) - f(a_0)| < \varepsilon/2M$$
$$b_0 - a_0 < \delta_g \Rightarrow |g(b_0) - g(a_0)| < \varepsilon/2M$$
Take $\delta = \min\{\delta_f, \delta_g\}$. Then, $b_0 - a_0 < \delta$ gives

$$\begin{aligned}
|(fg)(b_0) - (fg)(a_0)| &= |f(b_0)g(b_0) - f(a_0)g(a_0)| \\
&= |f(b_0)g(b_0) - [f(b_0) - c]\,g(a_0)| \quad \text{for some } |c| < \varepsilon/2M \\
&= |f(b_0)(g(b_0) - g(a_0)) + c\,g(a_0)| \\
&\le |f(b_0)|\,|g(b_0) - g(a_0)| + |c|\,|g(a_0)| \\
&< M \quad \varepsilon/2M \quad\quad \varepsilon/2M \quad M \\
&= \varepsilon.
\end{aligned}$$

Now, assume the result is true for unions of at most N disjoint intervals. Claim: also true for $N+1$.
for $\varepsilon > 0$, $\exists$ δ_N s.t. for any union of disjoint intervals
$(a_1, b_1), \ldots, (a_n, b_n)$ on $[a,b]$ with $n \le N$ we have
$$\Sigma_1^n\, b_i - a_i < \delta_N \Rightarrow \Sigma_1^n |(fg)(b_i) - (fg)(a_i)| < \varepsilon/2.$$
Define $\delta = \delta_N$. Then for any union of disjoint intervals
$(a_1, b_1), \ldots, (a_{N+1}, b_{N+1})$, with $\Sigma_1^{N+1} b_i - a_i < \delta$, we have
$\Sigma_1^N b_i - a_i < \delta$ and $b_{N+1} - a_{N+1} < \delta$, giving
$$\begin{aligned}
\Sigma_1^{N+1} |(fg)(b_i) - (fg)(a_i)| &= \Sigma_1^N |(fg)(b_i) - (fg)(a_i)| + |(fg)(b_{N+1}) - (fg)(a_{N+1})| \\
&< \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon.
\end{aligned}$$
$\square$

09 Sp. 2

(b) (fg) abs. cont. $\Longrightarrow$ $\int_{[a,b]} (fg)' \, dx = (fg)(b) - (fg)(a)$

$$\Longrightarrow \int_{[a,b]} fg' + f'g \, dx = f(b)g(b) - f(a)g(a)$$

$$\Longrightarrow \int_{[a,b]} fg' dx = f(b)g(b) - f(a)g(a) - \int_{[a,b]} f'g \, dx. \qquad \square$$

(c) Let f = devil's staircase let $g = 1$.

Then $f(0) = 0$, $f(1) = 1$, but $f' = 0$ a.e.

Thus,

$$\int_{[0,1]} fg' dx = \int_{[0,1]} f \cdot 0 \, dx = 0$$

But

$$\begin{aligned} & f(1)g(1) - f(0)g(0) - \int_{[0,1]} f'g \, dx \\ = \; & 1 \cdot 1 - 0 \cdot 1 - \int_{[0,1]} 0 \cdot 1 \, dx \\ = \; & 1 \end{aligned}$$

$\square$

09 Sp. 3

$f: \mathbb{R} \to \mathbb{R}$ integrable, $f = 0$ outside $[-1, 1]$.
Define $f_n(x) = f(x + 1/n)$. Must $f_n \to f$ in measure?

$f_n \to f$ in measure if $\mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) \to 0$ as $n \to \infty$, for a given $\varepsilon > 0$.

First, consider when $f = \chi_{(a,b)}$ for $(a,b) \subset [0,1]$.
$f_n \to f$ in L^1 $\Rightarrow$ $f_n \to f$ in measure.
Indeed:
$$\int |f_n - f| \, dx = \int |f(x + 1/n) - f(x)| \, dx$$
$$= \int |\chi_{(a - 1/n, b - 1/n)} - \chi_{(a,b)}| \, dx$$
$$= \int \chi_{(a - \frac{1}{n}, a)} + \chi_{(b - \frac{1}{n}, b)} \, dx, \quad \text{for large } n$$
$$= 2/n$$
which $\to 0$ as $n \to \infty$.

Now, for arbitrary f integrable,
Since we are using Lebesgue measure, for any $\varepsilon > 0$, we can find a simple function $\phi(x) \leq f(x)$ s.t.
$$\int |f - \phi| \, dx < \varepsilon/3.$$
and s.t. ϕ is defined on finitely many intervals $\{(a_i, b_i)\}_1^M$.
Define $\phi_n(x) = \phi(x + 1/n)$. Then $\int |f_n - \phi_n| \, dx < \varepsilon/3$ $\forall n$.
And $\phi_n \to \phi$ in L^1 since ϕ is the sum of M indicator functions on intervals, which we know already converge in L^1. $\Rightarrow \int |\phi_n - \phi| \, dx \to 0$ as $n \to \infty$.

Thus, $\exists N$ s.t. $\forall n \geq N$, $\int |\phi_n - \phi| \, dx < \varepsilon/3$.

Hence, $\forall n \geq N$,
$$\int |f_n - f| \, dx \leq \int |f_n - \phi_n| + |\phi_n - \phi| + |\phi - f| \, dx$$
$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3$$
$$= \varepsilon.$$

$\Rightarrow f_n \to f$ in L^1 $\Rightarrow$ $f_n \to f$ in measure.

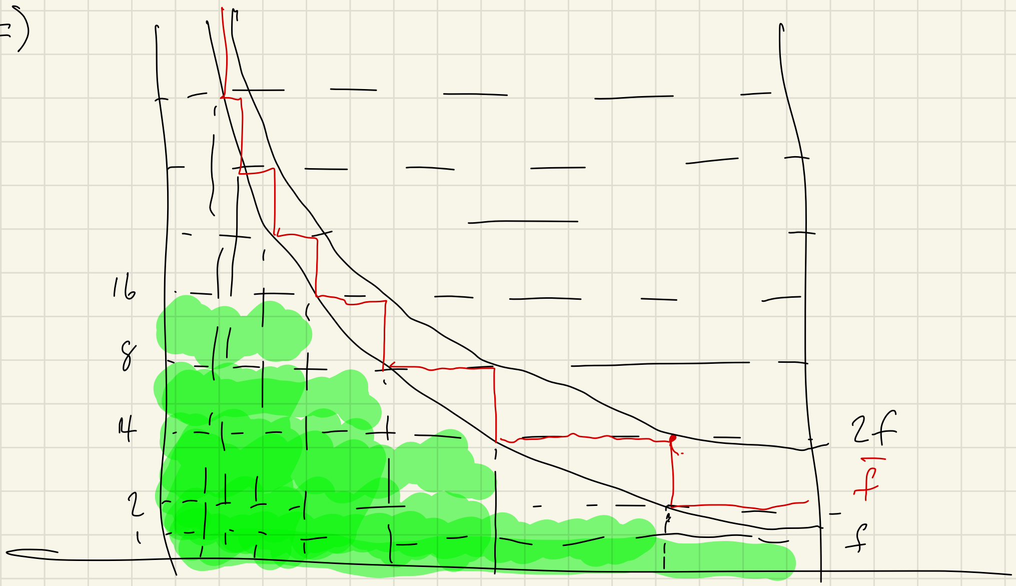
$\square$

09 Sp. 4

$\mu(X) < \infty$. $f \geq 0$ measurable on X.

Claim: f μ-integrable $\iff \sum_{n=0}^{\infty} 2^n \mu(\{x : f(x) \geq 2^n\}) < \infty$.

($\Leftarrow$)



Define $\bar{f} = 1 + \sum_{n=0}^{\infty} 2^n \chi_{\{x : f(x) \geq 2^n\}}$

We see that for any $x \in X$, we have $2^m \leq f(x) \leq 2^{m+1}$

for some $m = 0, 1, 2, \ldots$

Then $\bar{f}(x) = 1 + \sum_{n=0}^{m} 2^m = 1 + 2^{m+1} - 1 = 2^{m+1}$

$\Rightarrow$ $f(x) \leq \bar{f}(x)$ $\forall x \in X$.

Thus, $\int f(x) dx \leq \int \bar{f}(x) dx = \mu(X) + \sum_{n=0}^{\infty} 2^n \mu(\{x : f(x) \geq 2^n\}) < \infty$

which is to say f is μ-integrable.

($\Rightarrow$) for $x \in X$ with $2^m \leq f(x) \leq 2^{m+1}$, $\bar{f}(x) = 2^{m+1}$

$\Rightarrow$ $\bar{f}(x) \leq 2 f(x)$ $\forall x \in X$

$\Rightarrow$ $\sum_{n=0}^{\infty} 2^n \mu(\{x : f(x) \geq 2^n\}) = \int \bar{f}(x) dx - \mu(X) \leq 2 \int f(x) dx - \mu(X) < \infty$.

$\square$