$\boxed{\text{Spring 09}}$:

(1.) $(X, \mathcal{B}, \mu)$ meas. space, μ finite, $\mathcal{A} \subseteq \mathcal{B}$ algebra.

[$E \in \mathcal{B}$ is approx from inside by $\mathcal{A}$ if for $\varepsilon > 0$, $\exists A \in \mathcal{A}$ s.t. $\mu(E \setminus A) < \varepsilon$]

(a) Show: $\mathcal{C} = \{E \in \mathcal{B} : E \text{ approx inside by } \mathcal{A}\}$ is closed under countable unions

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Let $(E_i) \subseteq \mathcal{C}$ and let $A_i \in \mathcal{A}$ be such that $\mu(E_i \setminus A_i) < \frac{\varepsilon}{2^i}$

Finite unions: $\mu\left(\bigcup_{i=1}^k E_i \setminus \bigcup_{i=1}^k A_i\right) \leq \mu\left(\bigcup_{i=1}^k E_i \setminus A_i\right)$

$\leq \sum_{i=1}^k \mu(E_i \setminus A_i) = \sum_{i=1}^k \frac{\varepsilon}{2^i} < \varepsilon$; since $\mathcal{A}$ is an algebra, $\bigcup_{i=1}^k A_i \in \mathcal{A}$, hence $\bigcup_{i=1}^k E_i \in \mathcal{C}$.

Countable unions: Let $B_n = A_n \setminus \left(\bigcup_{j=1}^{n-1} A_j\right)$; hence $\bigcup B_i = \bigcup A_i$

$$\bigcup_{i=1}^\infty E_i \setminus \bigcup_{j=1}^\infty A_j \subseteq \bigcup_{i=1}^\infty E_i \setminus \bigcup_{j=1}^k A_j$$

$$= \left(\bigcup_{i=1}^k E_i \setminus \bigcup_{j=1}^k A_j\right) \cup \left(\bigcup_{i=k+1}^\infty E_i \setminus \bigcup_{j=1}^k A_j\right)$$

$$\subseteq \left(\bigcup_{i=1}^k E_i \setminus A_i\right) \cup \left(\bigcup_{i=k+1}^\infty E_i \setminus \bigcup_{j=1}^k A_j\right)$$

and
$$\left(\bigcup_{i=k+1}^\infty E_i \setminus \bigcup_{j=1}^k A_j\right) = \left[\bigcup_{i=k+1}^\infty E_i \setminus \bigcup_{j=1}^\infty A_j\right] \cup \left[\left(\bigcup_{i=k+1}^\infty E_i \setminus \bigcup_{i=1}^k A_i\right) \cap \left(\bigcup_{i=k+1}^\infty A_i\right)\right]$$

since $A_i \subseteq E_i$ $\Rightarrow$
$$\left[\bigcup_{i=k+1}^\infty \left(E_i \setminus \bigcup_{j=1}^\infty A_j\right)\right] \cup \left[\left(\bigcup_{i=k+1}^\infty A_i\right) \setminus \left(\bigcup_{i=1}^k A_i\right)\right]$$

$$\subseteq \left[\bigcup_{i=k+1}^\infty (E_i \setminus A_i)\right] \cup \left[\bigcup_{i=k+1}^\infty B_i\right]$$

Now, the measure is finite, hence $\infty > \mu\left(\bigcup_{i=1}^\infty A_i\right) = \mu\left(\bigcup_{i=1}^\infty B_i\right)$ $= \sum_{i=1}^\infty \mu(B_i)$ since B_i's are disjoint, hence $\mu(B_i) \to 0$ as $i \to \infty$

So now:

$\mu\left(\bigcup E_i \setminus \bigcup A_j\right) \leq \mu\left(\bigcup_{i=1}^k E_i \setminus A_i\right) + \mu\left(\bigcup_{i=k+1}^\infty E_i \setminus A_i\right) + \sum_{i=k+1}^\infty \mu(B_i) <$

So now:

$$\mu\left(\bigcup_{i=1}^{\infty} E_i \setminus \bigcup_{i=1}^{k} A_i\right) \leq \mu\left(\bigcup_{i=1}^{k} E_i \setminus A_i\right) + \mu\left(\bigcup_{k+1}^{\infty} E_i \setminus \bigcup_{i=1}^{k} A_i\right)$$

$$\leq \mu \cdot \varepsilon + \mu\left(\bigcup_{i=k+1}^{\infty} E_i \setminus A_i\right) + \mu\left(\bigcup_{i=k+1}^{\infty} B_k\right)$$

$$\leq \varepsilon + \sum_{i=k+1}^{\infty} \mu(E_i \setminus A_i) + \sum_{i=k+1}^{\infty} B_k$$

$$\leq \varepsilon + \sum_{i=k+1}^{\infty} \frac{\varepsilon}{2^i} + \sum_{i=k+1}^{\infty} B_k$$

$$\leq 2\varepsilon + \sum_{i=k+1}^{\infty} B_k \longrightarrow 2\varepsilon + 0$$

as $k \longrightarrow \infty$.

Hence for k large enough,

$\mu\left(\bigcup_{i=1}^{\infty} E_i \setminus \bigcup_{i=1}^{k} A_i\right) < \varepsilon$, hence $\bigcup_{i=1}^{\infty} E_i \in \mathcal{C}$

since $\bigcup_{i=1}^{k} A_i \in \mathcal{A}$ for all k.

(b) Find an example where $\mathcal{C}$ not closed under complements.

(2.) f, g abs cont on $[a,b]$.

(a) Show fg is abs. cont.

(b) Show $\int_a^b f g' \, dx = f(b)g(b) - f(a)g(a) - \int_a^b f' g \, dx$

(c) Show (b) invalid if f, g only diff. a.e. (not abs. cont.)

(a) f, g abs cont, hence for $\varepsilon > 0$, $\exists \delta_1, \delta_2$ s.t.

$$\sum_{i=1}^{n} |y_i - x_i| < \min(\delta_1, \delta_2) := \delta \Rightarrow \sum_{i=1}^{n} |f(y_i) - f(x_i)| < \varepsilon \text{ and } \sum_{i=1}^{n} |g(y_i) - g(x_i)| < \varepsilon.$$

Now, since f, g are abs. cont. on closed interval $[a,b]$ they are also bdd, i.e. $\exists M$ s.t. $f(x) \le M$, $g(x) \le M$ for all x.

Now suppose $\sum |y_i - x_i| < \delta$ and then:

$$\sum_{i=1}^{n} |f(y_i)g(y_i) - f(x_i)g(x_i)|$$

$$= \sum_{i=1}^{n} |f(y_i)g(y_i) - f(y_i)g(x_i) + f(y_i)g(x_i) - f(x_i)g(x_i)|$$

$$\le \sum_{i=1}^{n} |f(y_i)g(y_i) - f(y_i)g(x_i)| + |f(y_i)g(x_i) - f(x_i)g(x_i)|$$

$$= \sum_{i=1}^{n} \left(|f(y_i)||g(y_i) - g(x_i)| + |g(x_i)||f(y_i) - f(x_i)|\right)$$

$$< \sum_{i=1}^{n} M|g(y_i) - g(x_i)| + M|f(y_i) - f(x_i)| = M\left(\sum_{i=1}^{n} |g(y_i) - g(x_i)| + \sum_{i=1}^{n} |f(y_i) - f(x_i)|\right)$$

$$= M(2\varepsilon)$$

Hence let $\delta' > 0$ be s.t. $\sum |y_i - x_i| < \delta' \Rightarrow \sum |f(y_i) - f(x_i)| < \frac{\varepsilon}{2M}$ and $\sum |g(y_i) - f(x_i)| < \frac{\varepsilon}{2M}$

$$\Rightarrow \sum |fg(y_i) - fg(x_i)| < 2M\left(\frac{\varepsilon}{2M}\right) = \varepsilon$$, hence fg abs. cont.

(b) Since fg is absolutely continuous, we have:

$$\int_a^b (fg)' \, dx = fg(b) - fg(a)$$

$$\Rightarrow \int_a^b (f'g + g'f) \, dx = f(b)g(b) - f(a)g(a)$$

$$\Rightarrow \int_a^b f g' \, dx = f(b)g(b) - f(a)g(a) - \int_a^b f'g \, dx$$

(3.) $f: \mathbb{R} \to \mathbb{R}$ int, $f = 0$ outside $[-1, 1]$.
Define $f_n(x) = f(x + \frac{1}{n})$. Must $f_n \to f$ in measure?

Recall that convergence in $L^1 \Rightarrow$ convergence in measure.

- Continuous case: Suppose f is continuous, i.e. for $\varepsilon > 0$, $\exists N$ s.t. $|x-y| < \frac{1}{N} \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{3}$. Now $n \geq N \Rightarrow \frac{1}{n} < \frac{1}{N}$, hence

$$|x - (x + \tfrac{1}{n})| = \tfrac{1}{n} < \tfrac{1}{N} \Rightarrow |f(x) - f(x + \tfrac{1}{n})| < \tfrac{\varepsilon}{3}$$

Then we have:

$$\|f - f_n\|_{L^1} = \int_{\mathbb{R}} |f - f_n| dx = \int_{\mathbb{R}} |f(x) - f(x + \tfrac{1}{n})| dx$$

$$\overset{*}{=} \int_{-1-\frac{1}{n}}^{1} |f(x) - f(x + \tfrac{1}{n})| < \int_{-1-\frac{1}{n}}^{1} \frac{\varepsilon}{3} dx = \frac{\varepsilon}{3}\left(1 - (-1 - \tfrac{1}{n})\right) = \frac{\varepsilon}{3}(2 + \tfrac{1}{n}) \leq \varepsilon$$

for all $n \geq N$

$*$ since $f(x) = 0$ for $x \notin [-1, 1]$

Hence $n > N \Rightarrow \|f - f_n\|_{L^1} < \varepsilon$, i.e. $f_n \to f$ in L^1

- General case: Since the measure is Lebesgue, we have that continuous functions with bdd support are dense in L^1.

So for f integrable, $\exists$ cont. g s.t. $\int |g - f| < \frac{\varepsilon}{3}$

Then, for $n \geq N$:

$$\|f - f_n\|_{L^1} = \int |f - f_n| dx = \int |f - g + g - g_n + g_n - f_n| dx$$
$$\leq \int (|f - g| + |g - g_n| + |g_n - f_n|) dx$$
$$= \int |f - g| dx + \int |g - g_n| dx + \int |g_n - f_n| dx$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \int |g_n - f_n| dx$$

continuous case

Now recall that Lebesgue measure is translation invariant:

$$\int |g_n - f_n| = \int |g(x + \tfrac{1}{n}) - f(x + \tfrac{1}{n})| dx = \int |g(x) - f(x)| d(x + \tfrac{1}{n}) = \int |g - f|$$

Hence:

$$\|f - f_n\|_{L^1} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \int |g_n - f_n| dx = \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \int |g - f| dx < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Hence $\|f - f_n\|_{L^1} < \varepsilon \Rightarrow f_n \to f$ in $L^1 \Rightarrow f_n \to f$ in measure.

④ $\mu(X)<\infty$, $f \geq 0$ measurable

Show: f μ-integrable $\iff \sum_{n=0}^{\infty} 2^n \mu(\{f(x) \geq 2^n\}) < \infty$

($\Leftarrow$) Suppose $\sum_{n=1}^{\infty} 2^n \mu(\{f(x) \geq 2^n\}) < \infty$.

Let $A_n = \{2^n \leq f(x) < 2^{n+1}\}$, $A_{-1} = \{0 \leq f(x) < 1\}$

Since $f \geq 0$, $\bigcup_{i=-1}^{\infty} A_i = X$; then:

$$\int f\,d\mu = \int_{A_{-1}} f\,d\mu + \sum_{n=0}^{\infty} \int_{A_n} f\,d\mu \leq \int_{A_{-1}} d\mu + \sum_{n=0}^{\infty} \int_{A_n} 2^n\,d\mu$$

$$= \mu(A_{-1}) + \sum_{n=0}^{\infty} 2^n \mu(A_n) = \mu(A_{-1}) + \sum_{n=0}^{\infty} 2^n \mu(\{2^n \leq f(x) < 2^{n+1}\})$$

$$\leq \mu(A_{-1}) + \sum_{n=0}^{\infty} 2^n \mu(\{2^n \leq f(x)\}) < \infty$$

finite since $\mu(X)<\infty$ — finite by hypothesis

($\Rightarrow$) Suppose $\int_X f\,d\mu < \infty$

Then we have $\infty > \int_X f\,d\mu = \int_{\bigcup A_n} f\,d\mu = \int_X \left(\sum_{n=-1}^{\infty} \chi_{A_n} f\right) d\mu$

$$\overset{(=)}{=} \sum_{n=-1}^{\infty} \int_{A_n} f\,d\mu \geq \sum_{n=-1}^{\infty} 2^n \mu(A_n) \Rightarrow \sum_{n=-1}^{\infty} 2^n \mu(A_n) < \infty$$

by Tonelli since all positive (or MCT)

Now consider the sets $B_n = \{2^n \leq f(x)\}$

Clearly $B_n = \bigcup_{j=n}^{\infty} A_j$

Now: $\sum_{n=0}^{\infty} 2^n \mu(\{f(x) \geq 2^n\}) = \sum_{n=0}^{\infty} 2^n \mu(B_n) = \sum_{n=0}^{\infty} 2^n \mu\left(\bigcup_{j=n}^{\infty} A_j\right)$

$$\overset{(=)}{=} \sum_{n=0}^{\infty} 2^n \left(\sum_{j=n}^{\infty} \mu(A_j)\right) = 2^0(A_0 + A_1 + \dots) + 2^1(A_1 + A_2 + \dots) + 2^2(A_2 + \dots)$$

A_j disjoint

$$= A_0 + (1+2)A_1 + (1+2+2^2)A_2 + \dots$$

$$= \sum_{j=0}^{\infty} \left(\sum_{n=0}^{j} 2^n \mu(A_j)\right)$$

Lemma: let $s = \sum_{n=0}^{j} 2^n \Rightarrow 2s = \sum_{n=0}^{j} 2^{n+1}$ and $s = 2s - s = \sum_{n=0}^{j} 2^{n+1} - \sum_{n=0}^{j} 2^n$

$= (2 + 2^2 + \dots 2^{j+1}) - (1 + 2 + \dots + 2^j) = 2^{j+1} - 1$

So now $\sum_{j=0}^{\infty} \left(\sum_{n=0}^{j} 2^n \mu(A_j)\right) = \sum_{j=0}^{\infty} (2^{j+1} - 1)\mu(A_j) = \sum_{j=0}^{\infty} 2^{j+1} \mu(A_j) - \sum_{j=0}^{\infty} \mu(A_j)$

Recall that $\int_X f\,d\mu < \infty \Rightarrow \sum_{j=0}^{\infty} 2^{j+1} \mu(A_j) < \infty$ and $\mu(X) < \infty$,

$$\Rightarrow \underbrace{\sum_{j=0}^{\infty} \mu(A_j) < \infty}_{A_i \text{ disjoint}} \Rightarrow \sum_{n=0}^{\infty} 2^n \mu(\{f \geq 2^n\}) = \sum_{j=0}^{\infty} 2^{j+1} \mu(A_j) - \sum_{j=0}^{\infty} \mu(A_j) < \infty$$